Deep Learning for Graphics

Machine Learning Introduction

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UCL	UCL/Facebook	UCL	Adobe Research	U Washington	UCL

Course Overview

- **Part I: Introduction and ML Basics**
- Part II: Supervised Neural Networks: Theory and Applications
- Part III: Unsupervised Neural Networks: Theory and Applications
- Part IV: Beyond Image Data

Machine Learning

Machine Learning

Machine learning is a field of **computer science** that uses statistical techniques to give computer systems the ability to ***learn*** (i.e., progressively improve performance on a specific task) with **data**, without being explicitly programmed.

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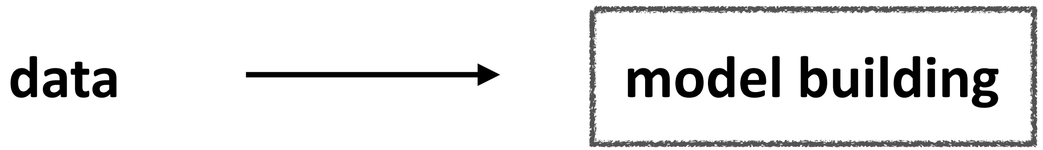

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Machine Learning Variants

- **Supervised**
  - Classification
  - Regression
  - Data consolidation
- **Unsupervised**
  - Clustering
  - Dimensionality Reduction
- **Weakly supervised/semi-supervised**

  Some data supervised, some unsupervised
- **Reinforcement learning**

  Supervision: sparse reward for a sequence of decisions

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Classification Examples

- Digit Recognition

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- Spam Detection

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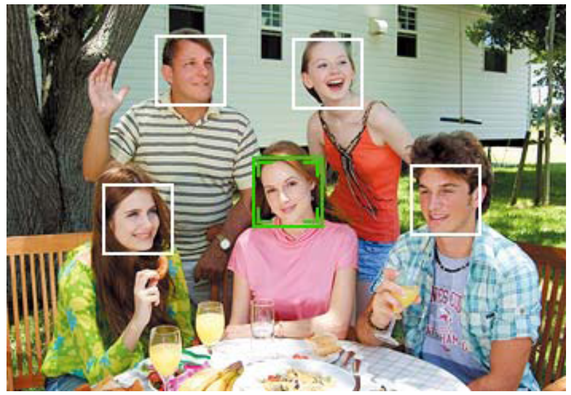
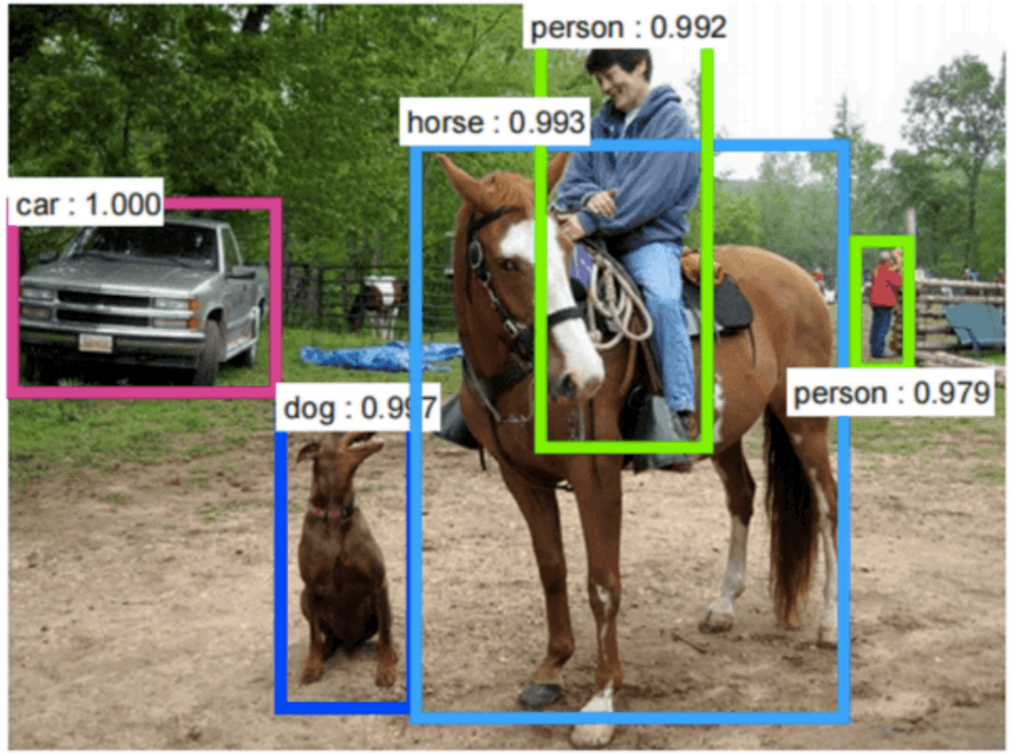
- Face detection



Segmentation + Classification in Real Images



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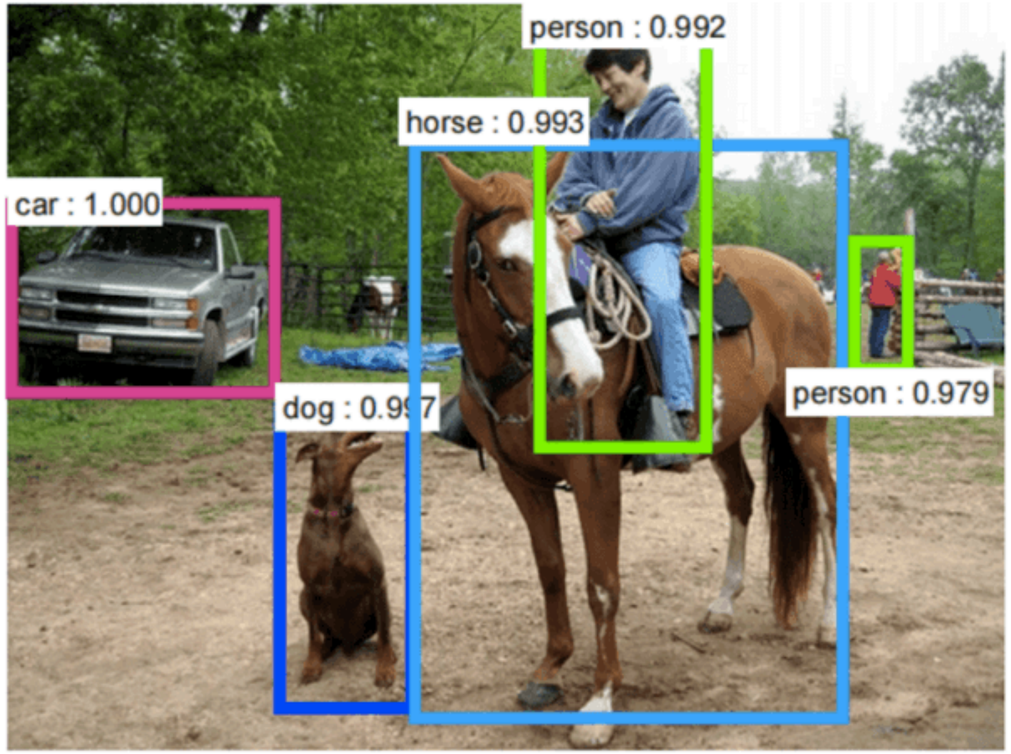

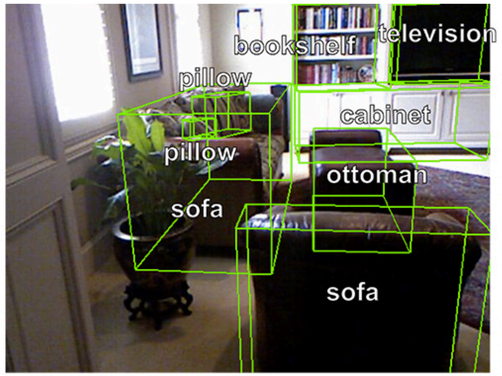

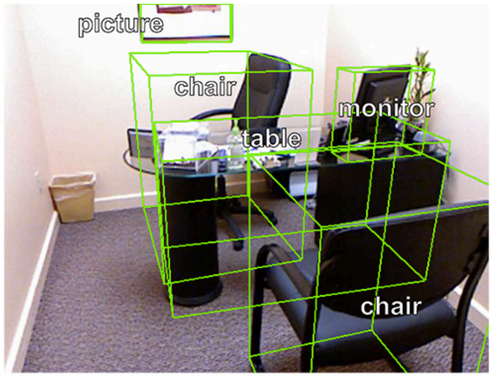

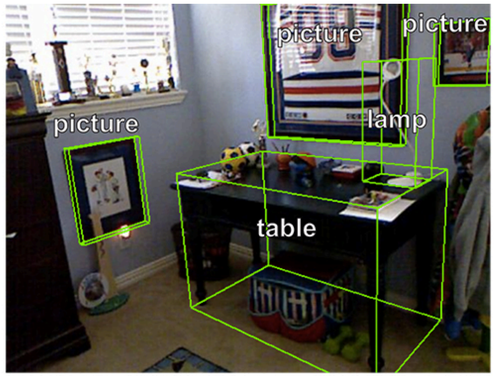

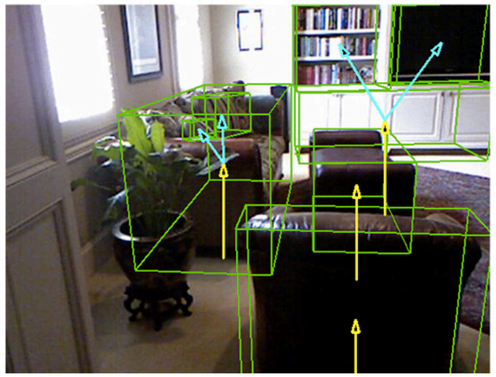

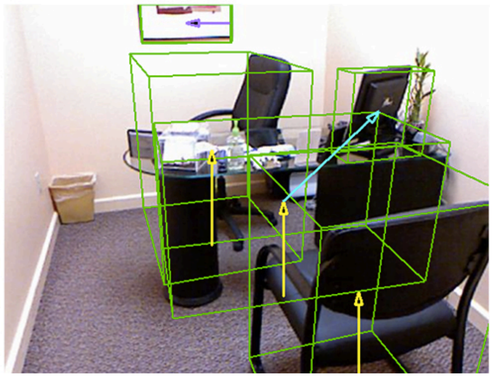

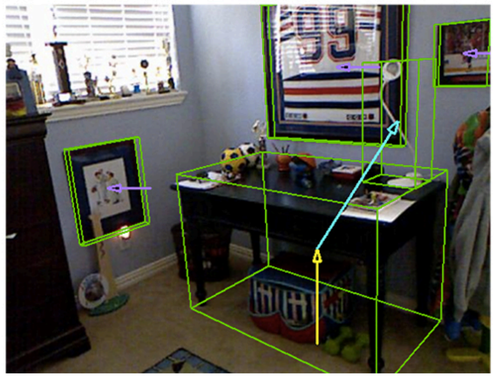

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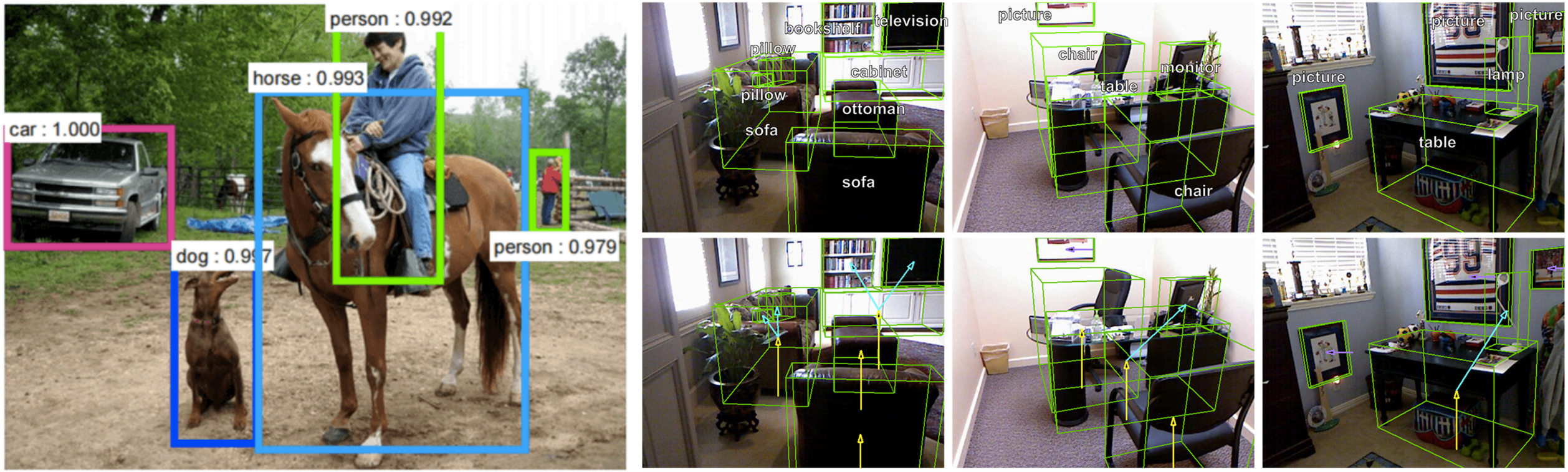

Evaluation measures: Confusion matrix, ROC curve, precision, recall, etc.

`Faceness' Function: Classifier

background

face

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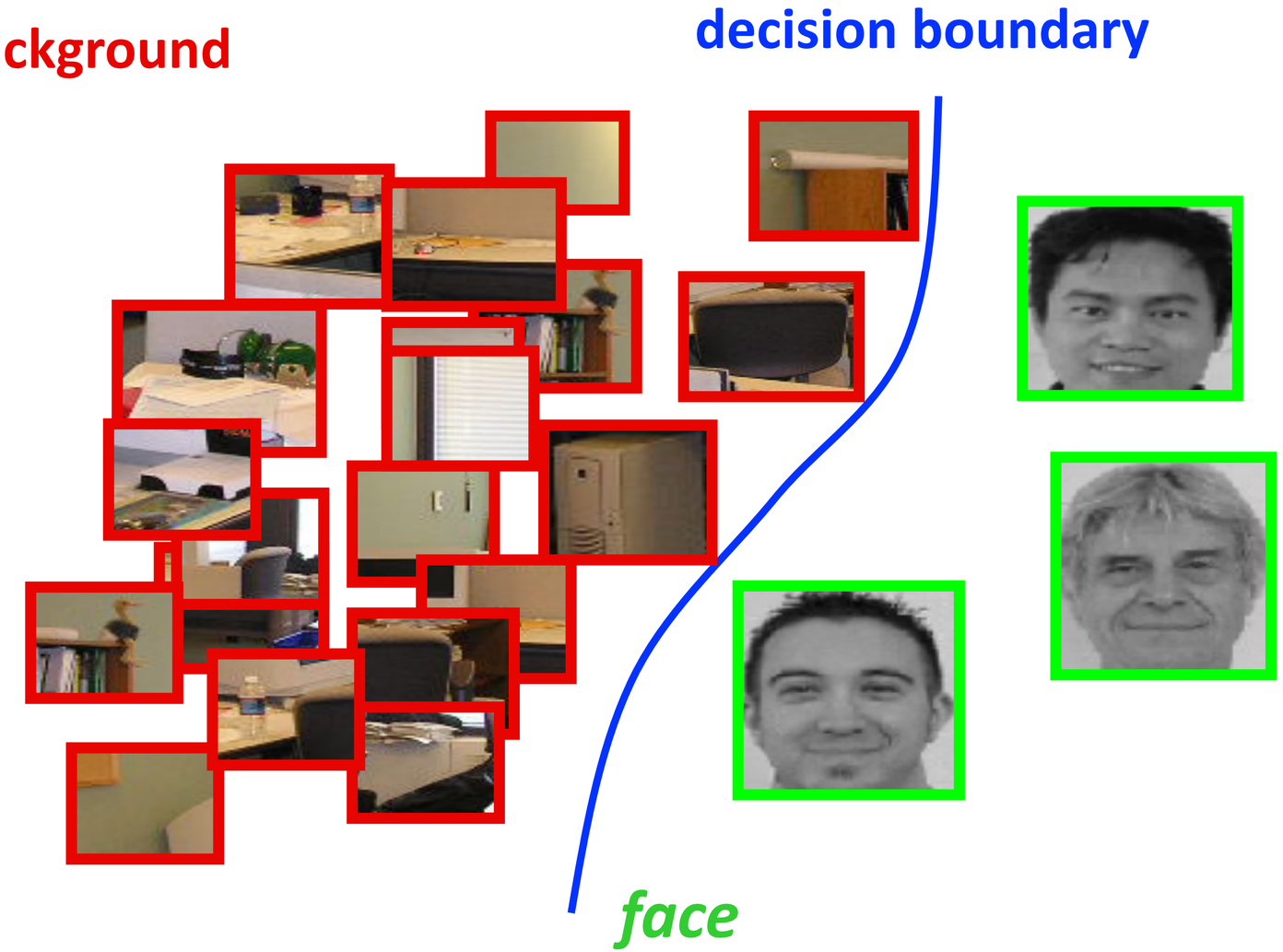
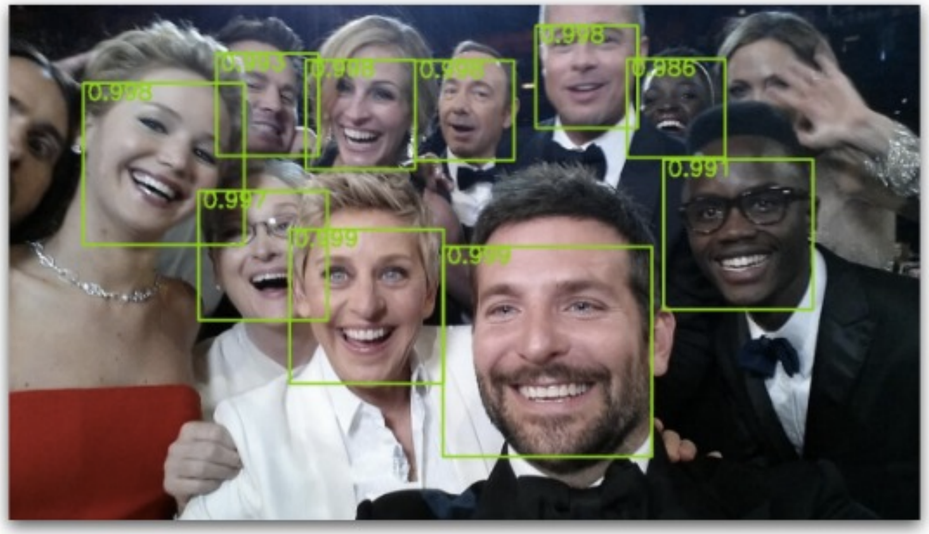


Face Detection



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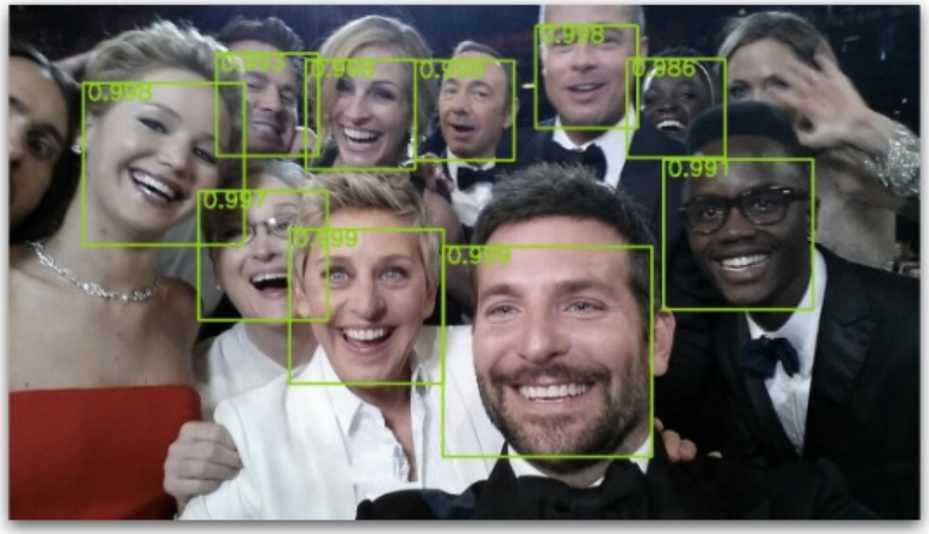

CMU Science Lab

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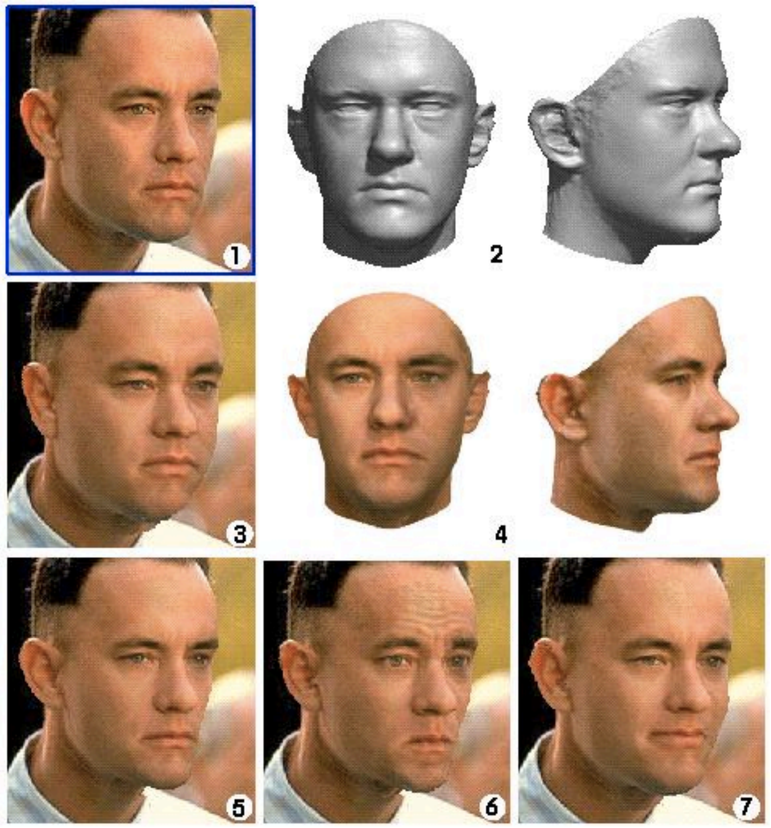
Human Face/Pose Estimation



[Blanz and Vetter, Siggraph, 1999]

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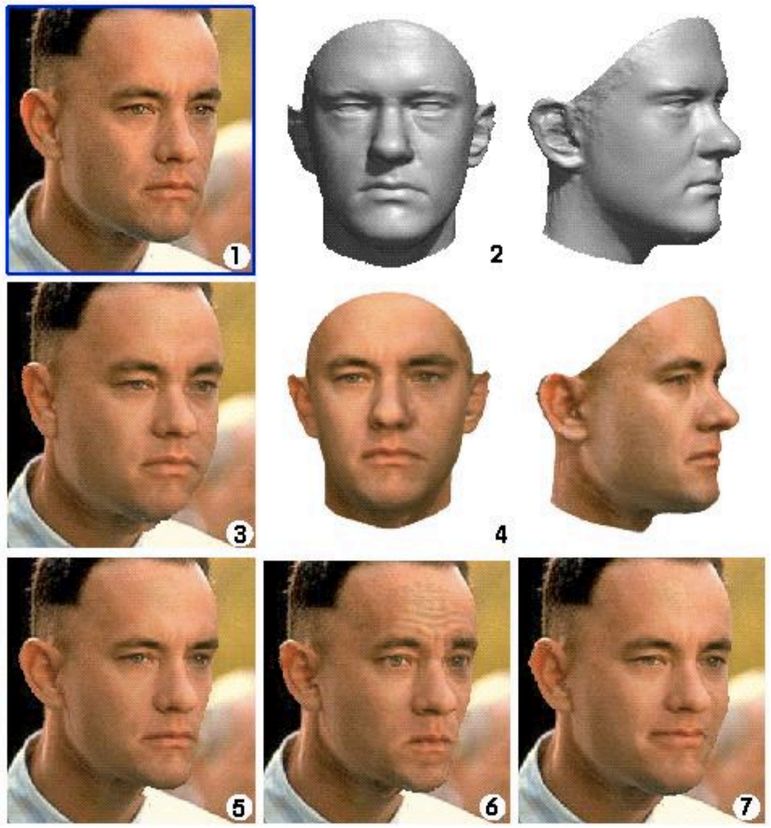

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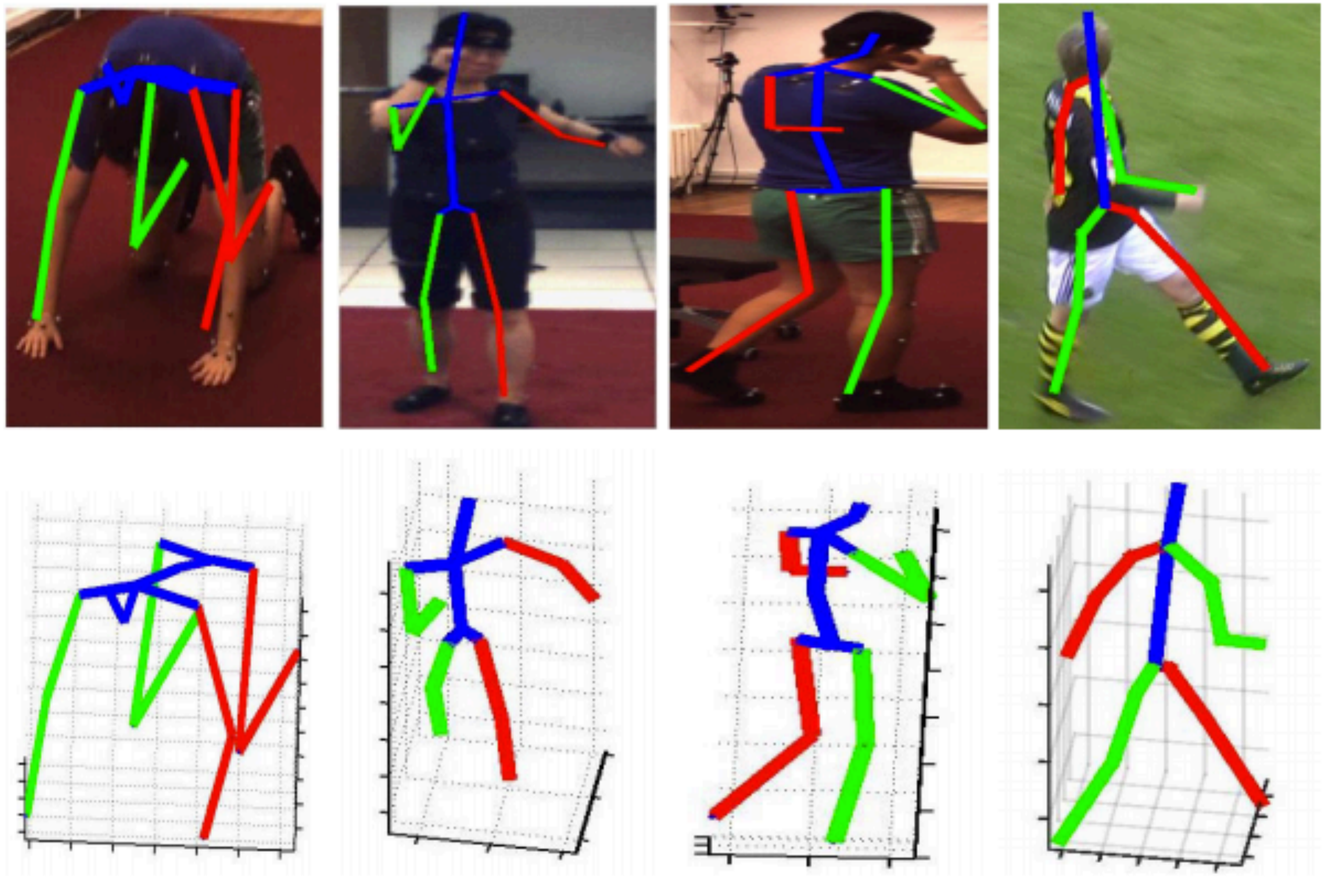
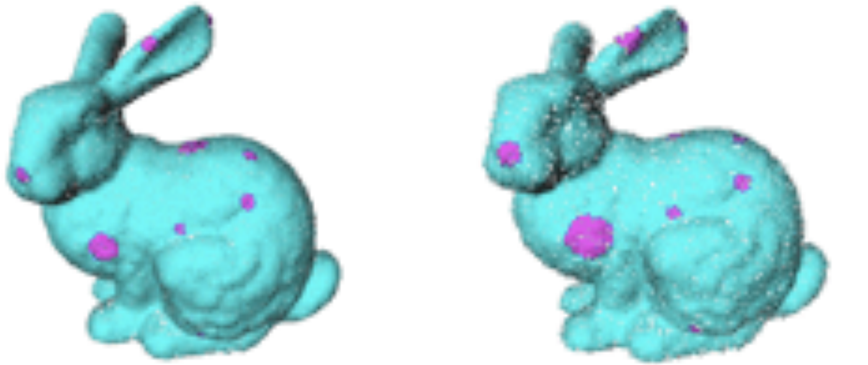
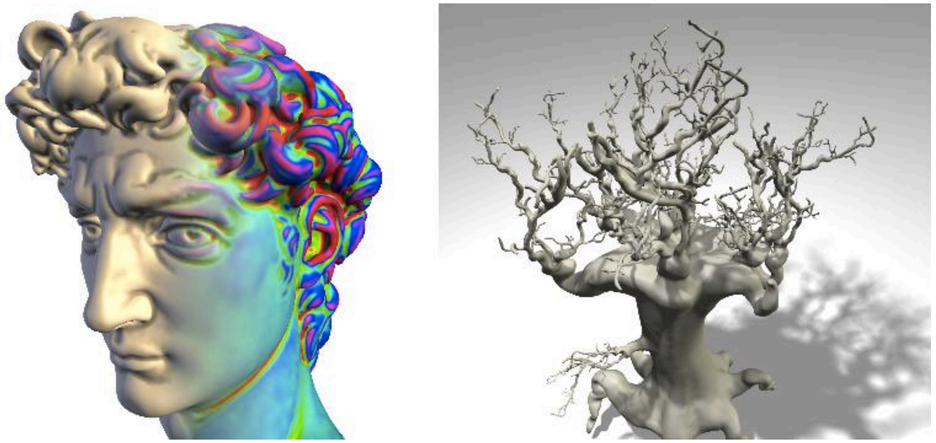


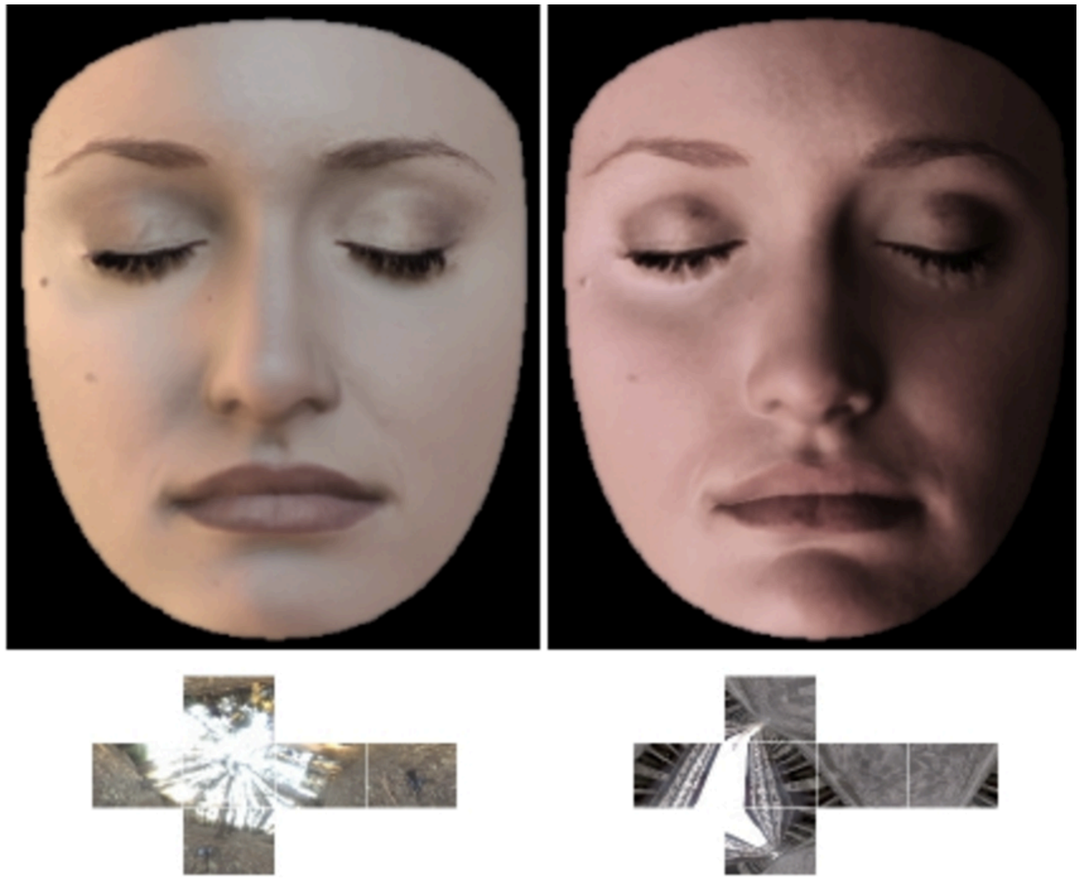
Regression: Model Estimation

[Mitra et al. SoCG, 2003]





[Guennebaud et al., Siggraph, 2007]



[Zwicker et al., EGSR, 2005]

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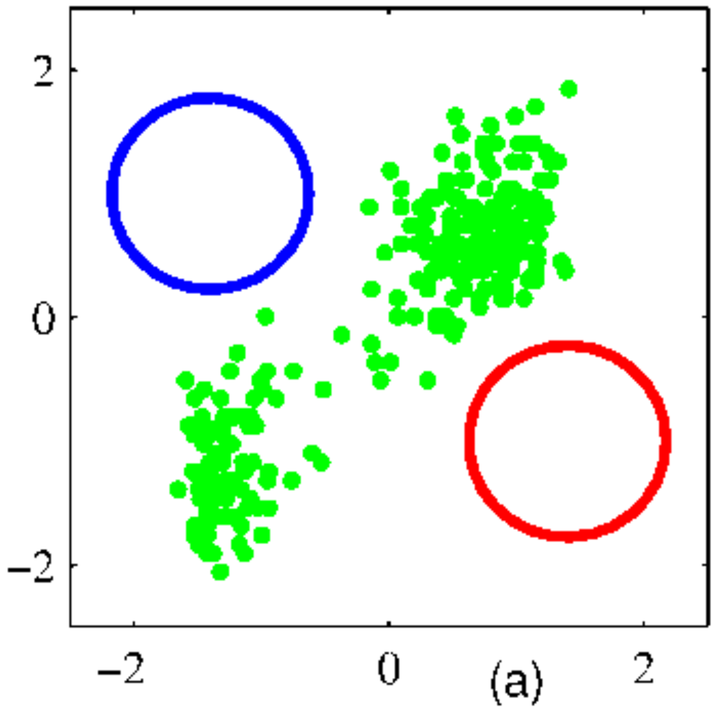
Clustering: Color Points According to X



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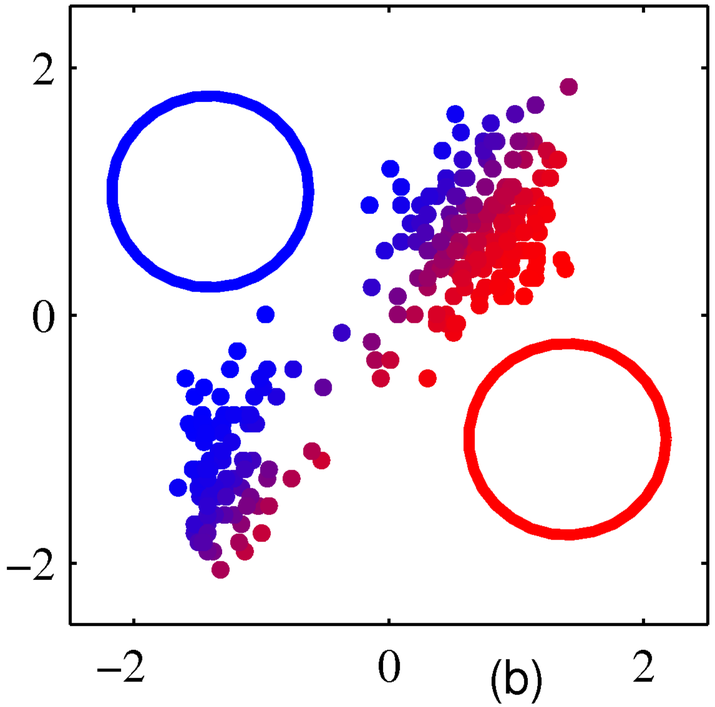

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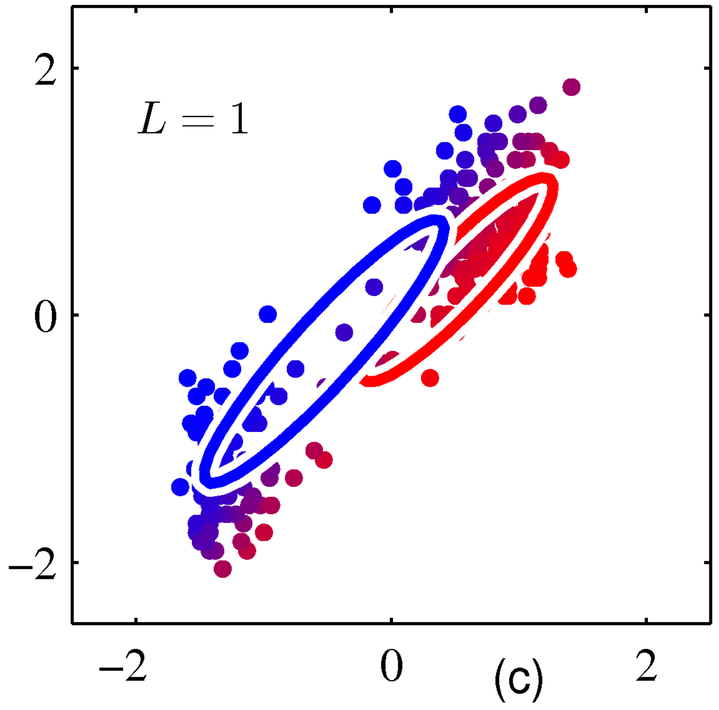

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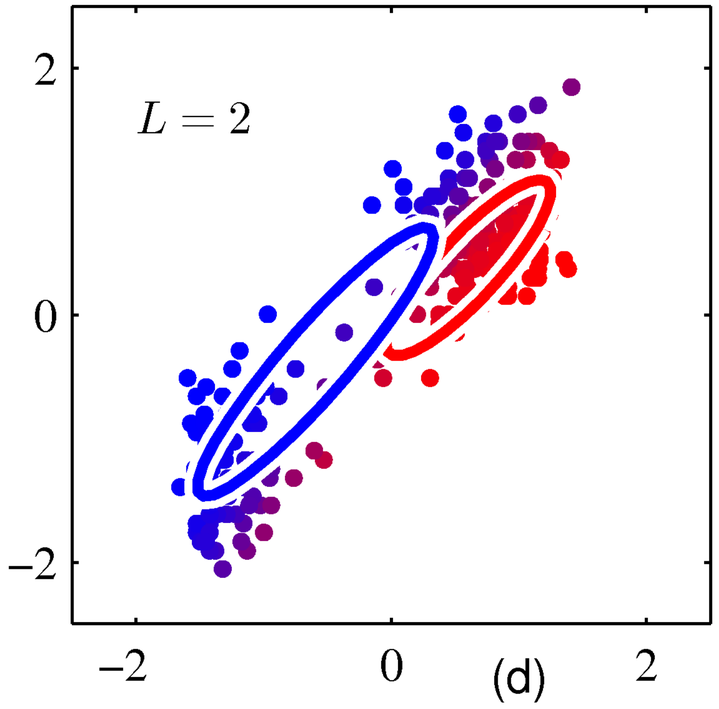

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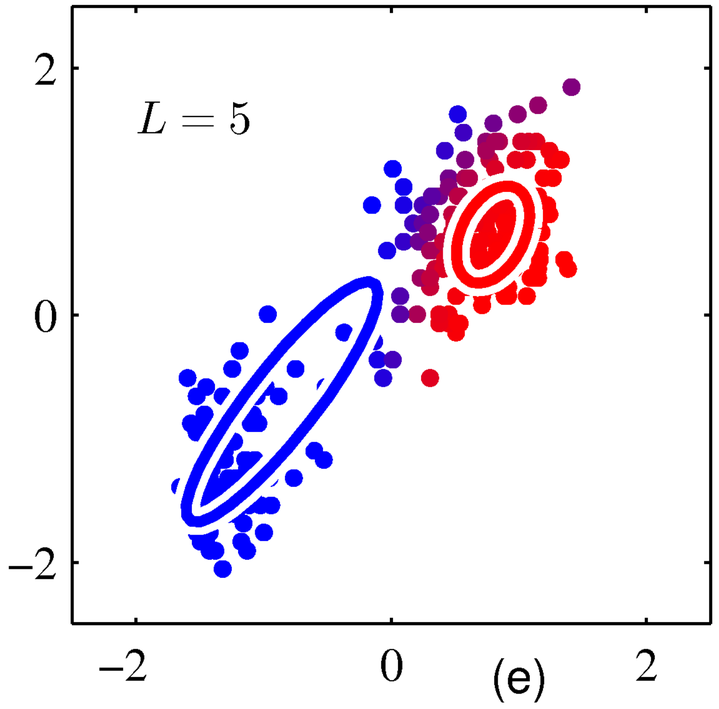

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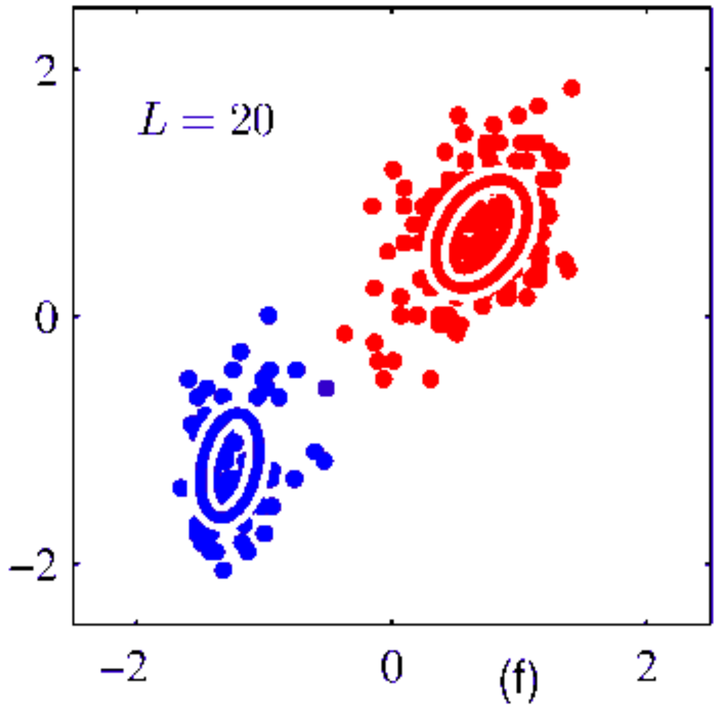
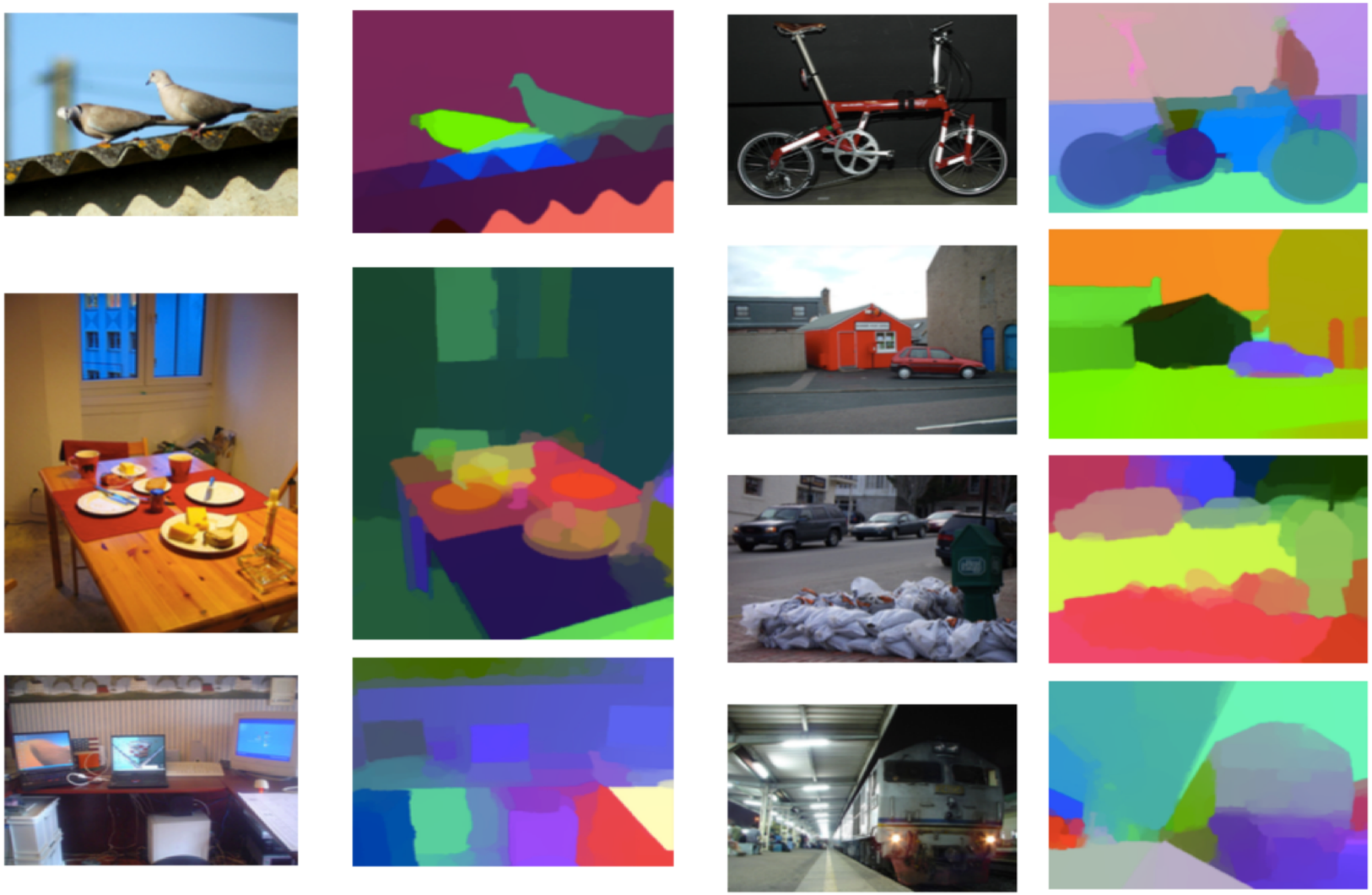
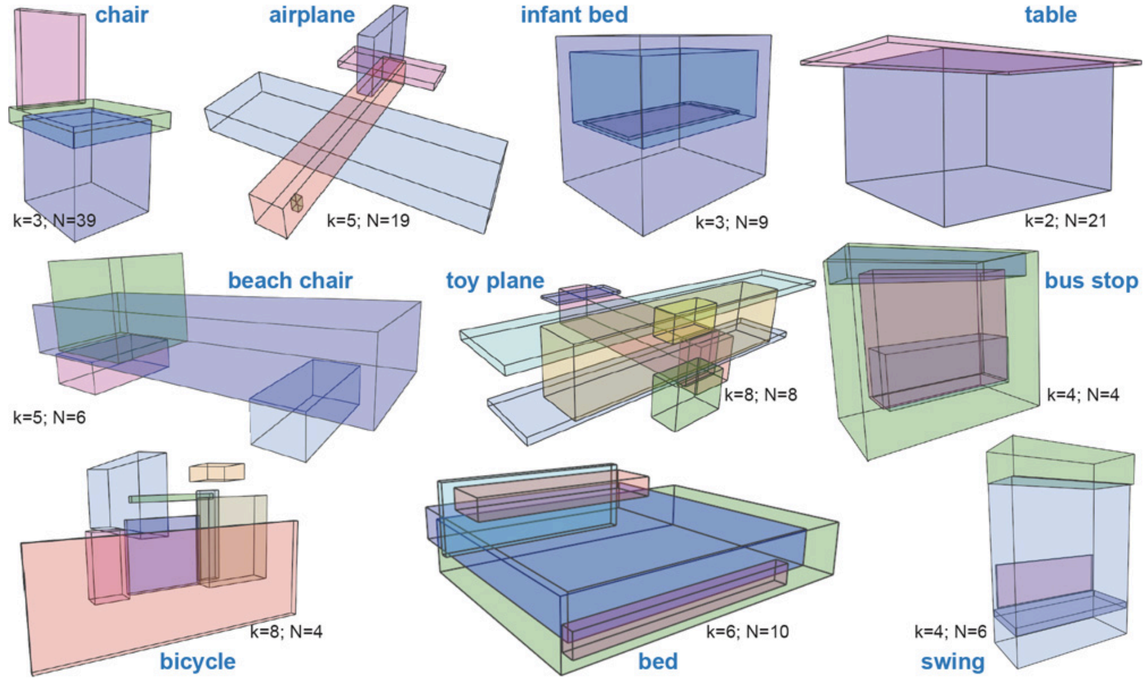


Clustering Examples: Image Segmentation using NCuts



Clustering Examples

[Chu et al., TVCG, 2009]



[Zheng et al., Eurographics, 2014]

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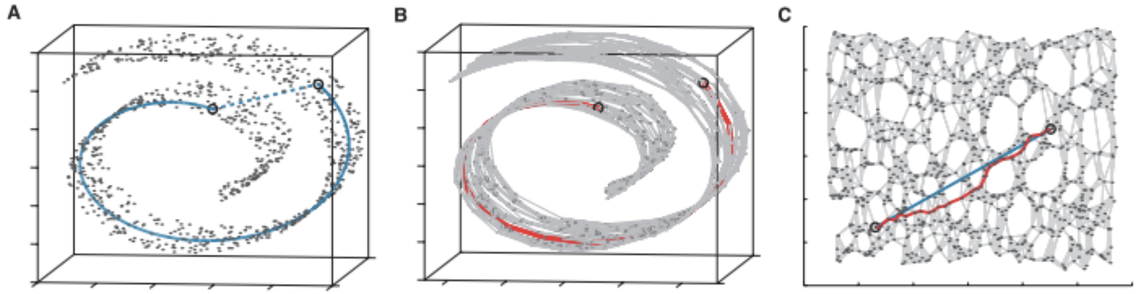
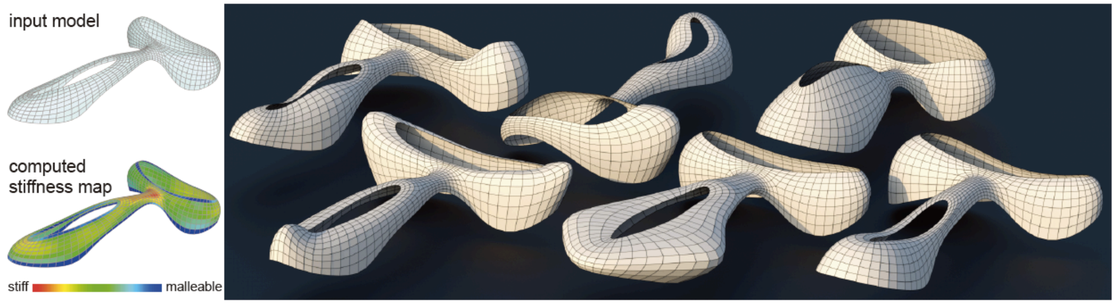
Dimensionality Reduction (Manifold Learning)

Isomap

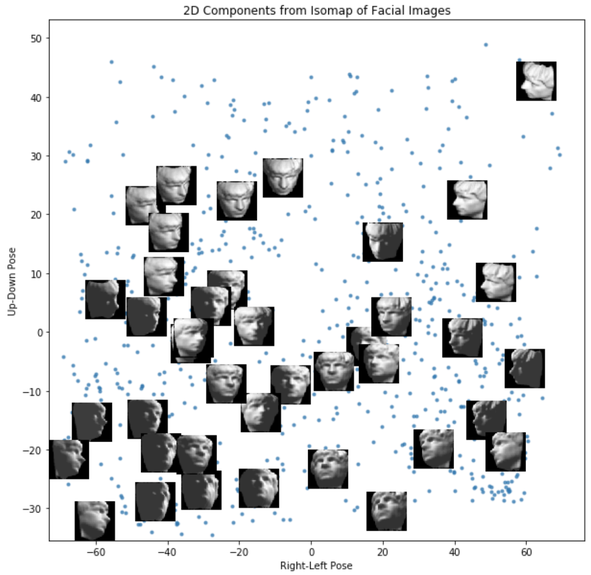


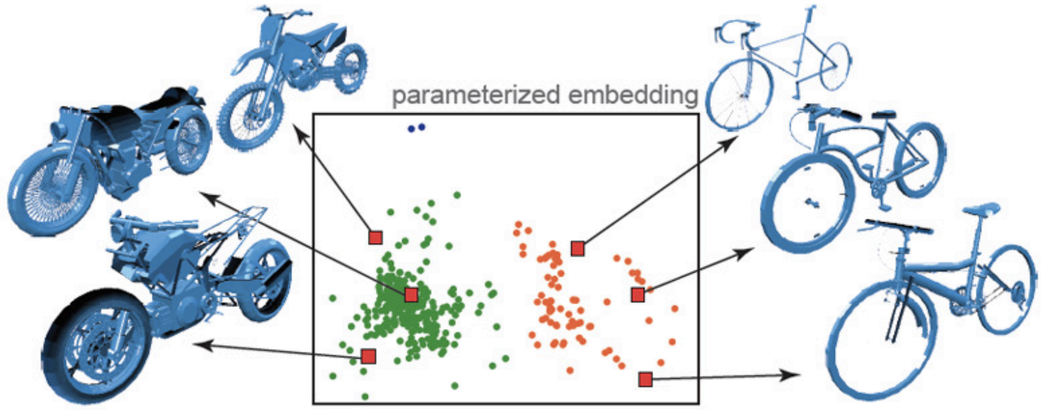
[Tenenbaum et al., Science, 2000]



[Yang et al., TOG, 2011]

Face Manifold





[Averkiou et al., Eurographics, 2014]

Example of Nonlinear Manifold: Faces

$\mathbf{x}_1$

$\mathbf{x}_2$

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$\mathbf{x}_1$

$\frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$

$\mathbf{x}_2$

Example of Nonlinear Manifold: Faces

X

$\mathbf{x}_1$

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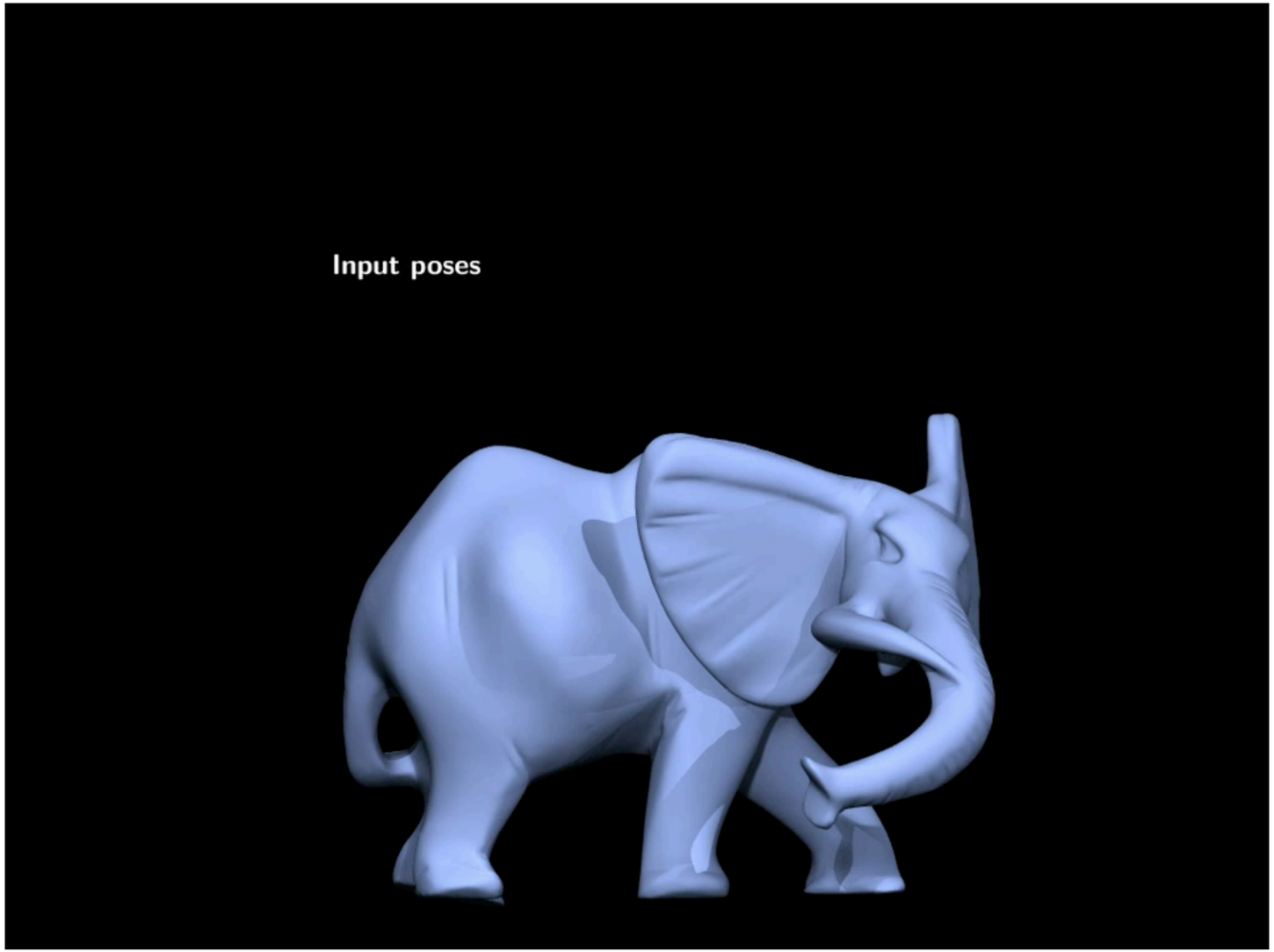
$\mathbf{x}_2$

Morphing (Interpolation in Shape Space)



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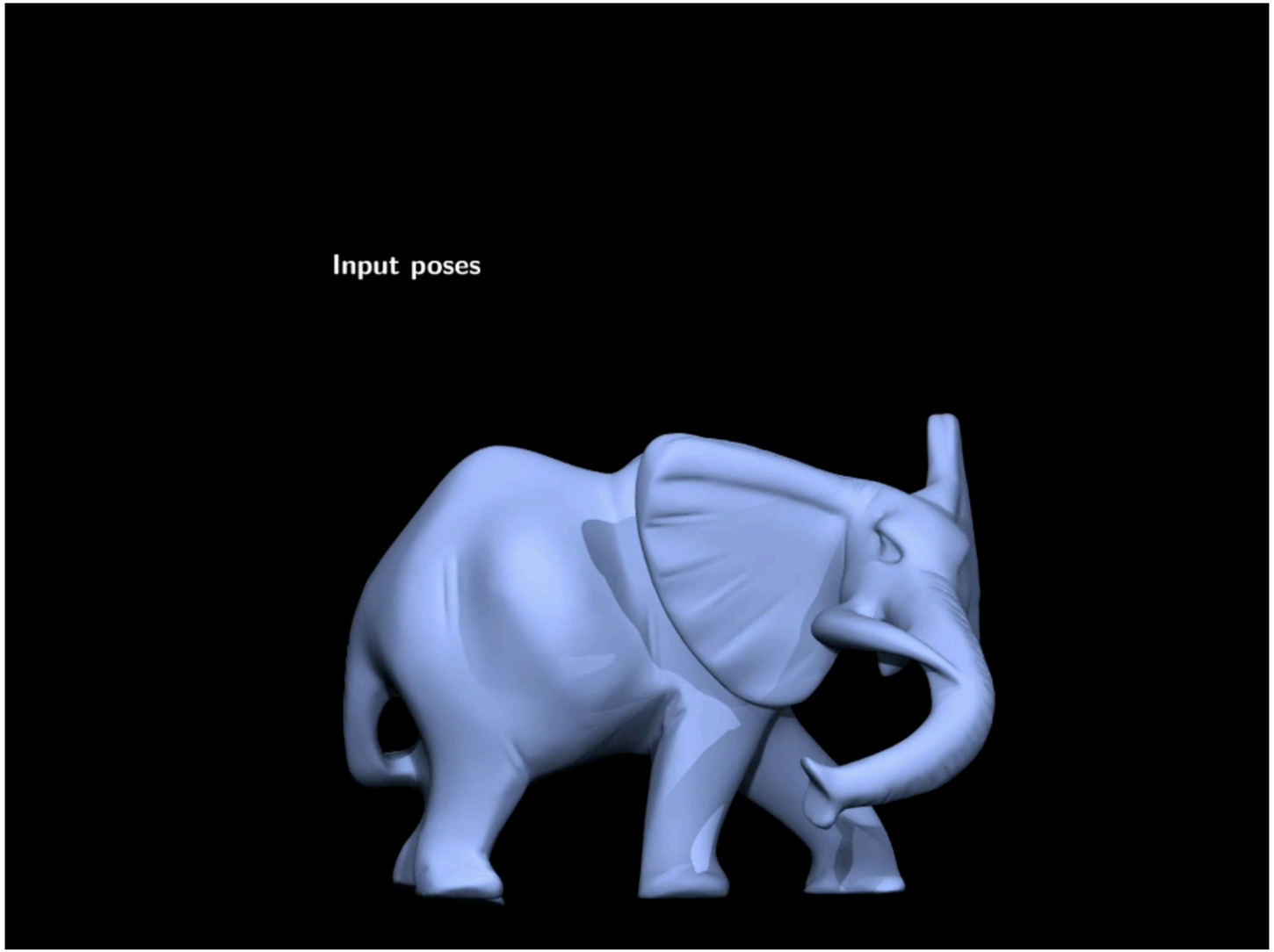

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Moving Along Learned Face Manifold

Trajectory along the “male” dimension

[Lample et. al. Fader Networks, NIPS 2017]

Moving Along Learned Face Manifold

Trajectory along the “male” dimension

Trajectory along the “young” dimension

[Lample et. al. Fader Networks, NIPS 2017]

Notations: Vectors and Matrices

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$\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta)$

Notations: Vectors and Matrices

- linear **independence**; **rank** of a matrix
- **span** of a matrix

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Notations: Vectors and Matrices (cont.)

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null space $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = 0\}$

Eigenvectors and Eigenvalues

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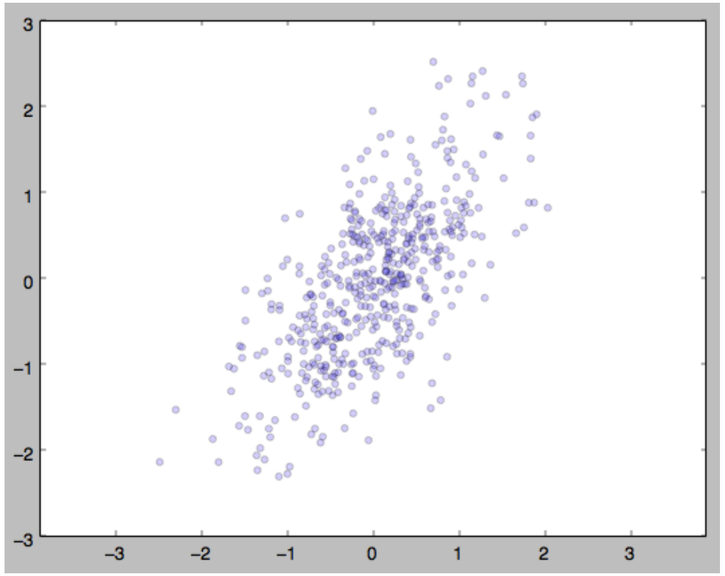
- All eigenvalues of symmetric matrices are real.
- Any real symmetric nxn matrix has a set of **n** mutually orthogonal eigenvectors.

Code Example



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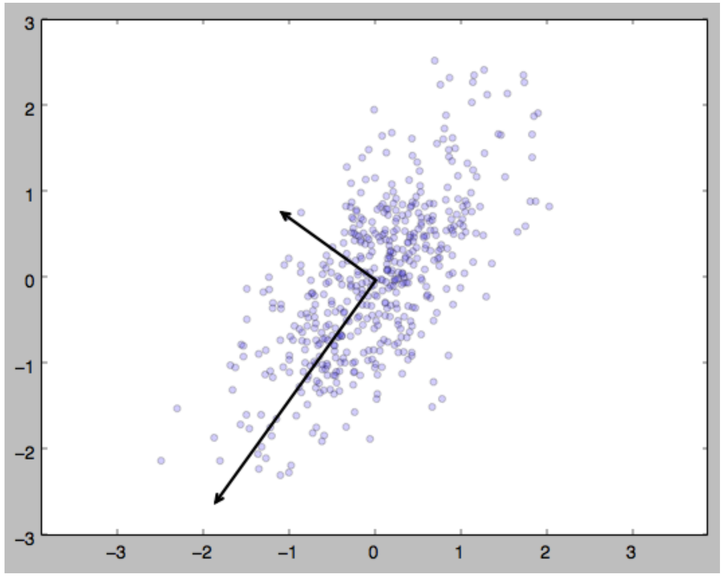

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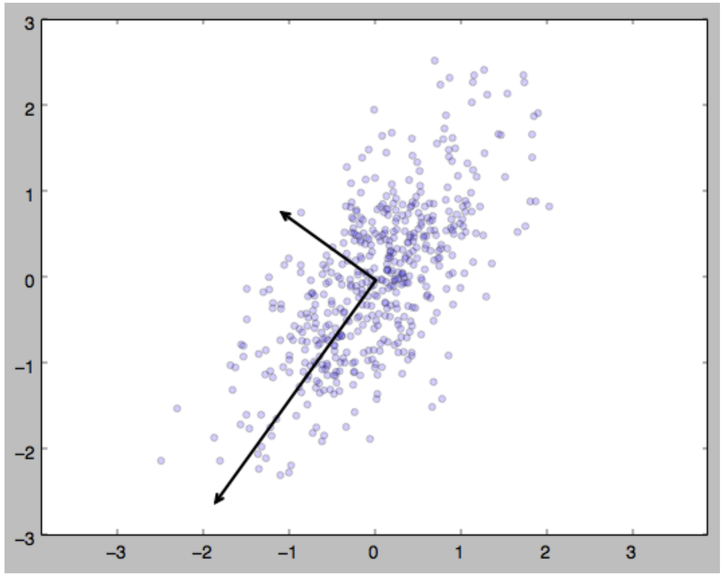


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rng = np.random.RandomState(10)
X = np.dot(rng.rand(2, 2), rng.randn(2, 500)).T

mean_vec = np.mean(X, axis=0)
cov_mat = (X - mean_vec).T.dot((X - mean_vec)) / (X.shape[0]-1)
eig_vals, eig_vecs = np.linalg.eig(cov_mat)
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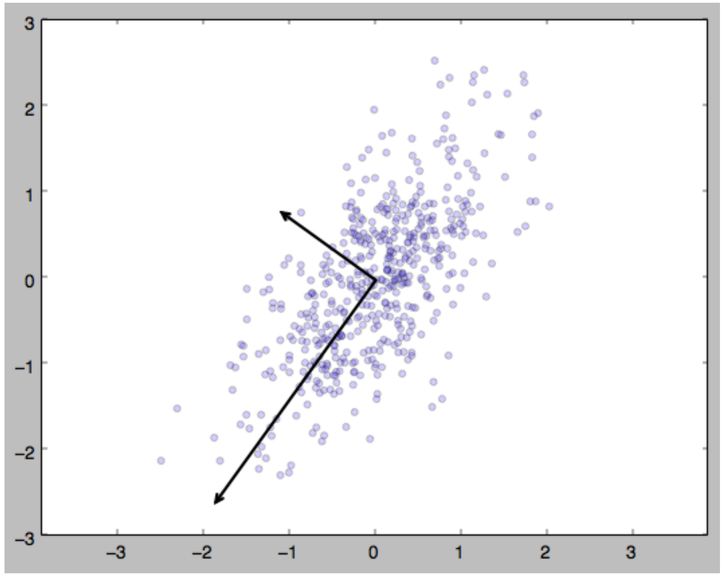


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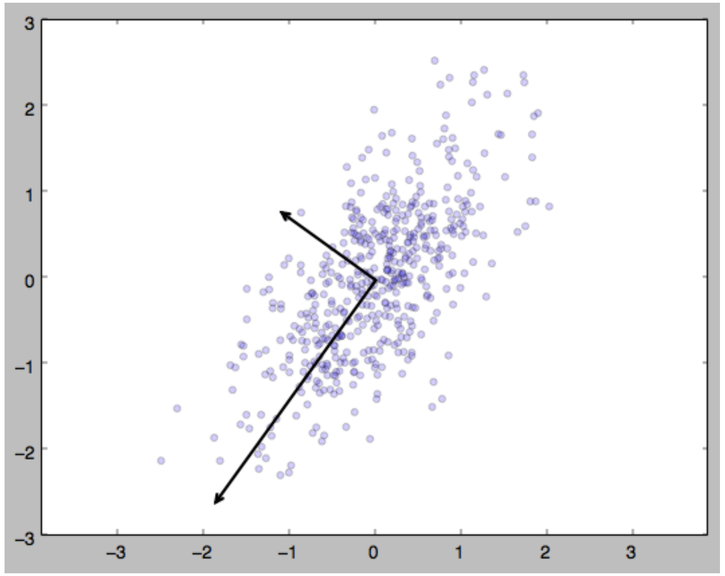
```

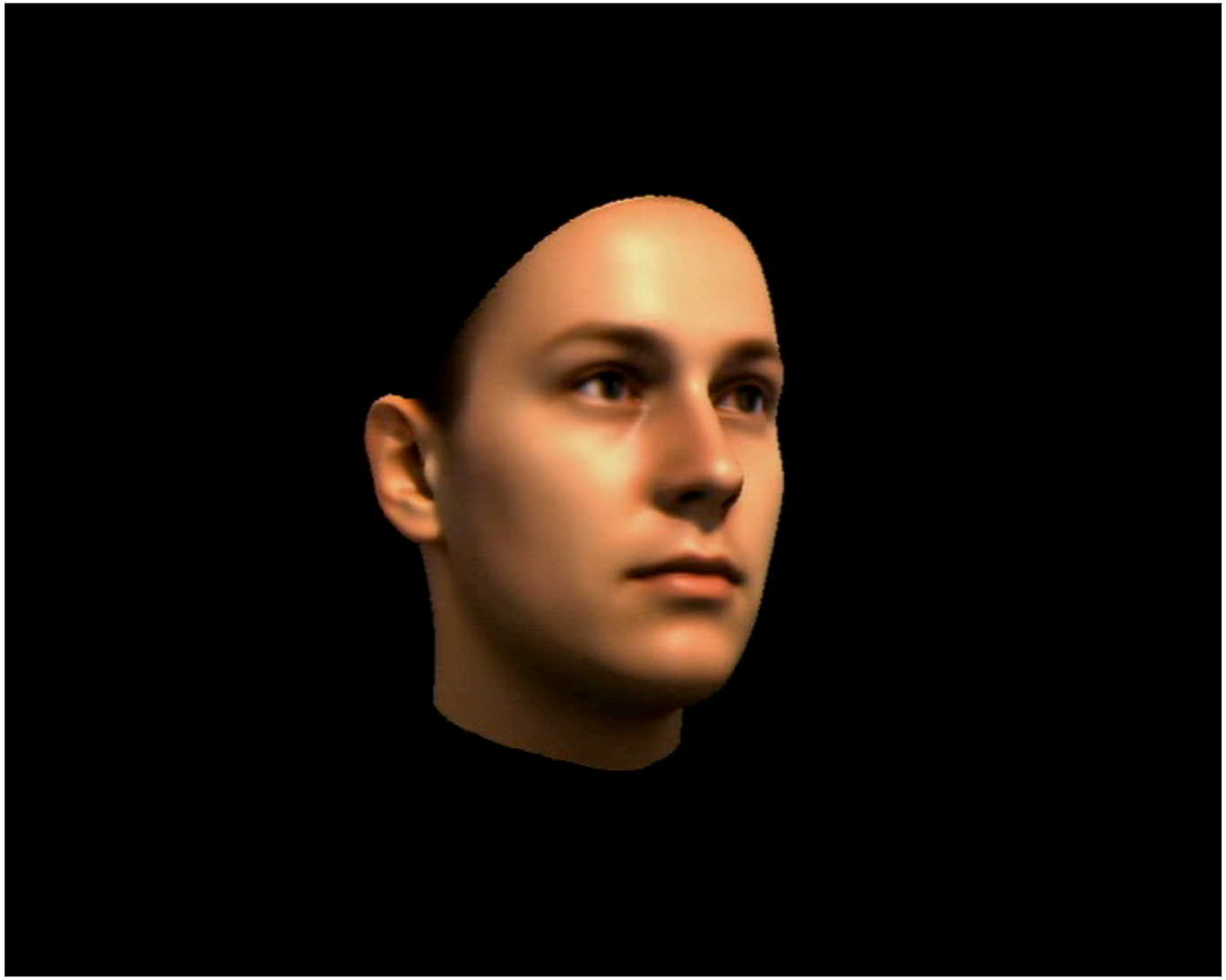
Code Example



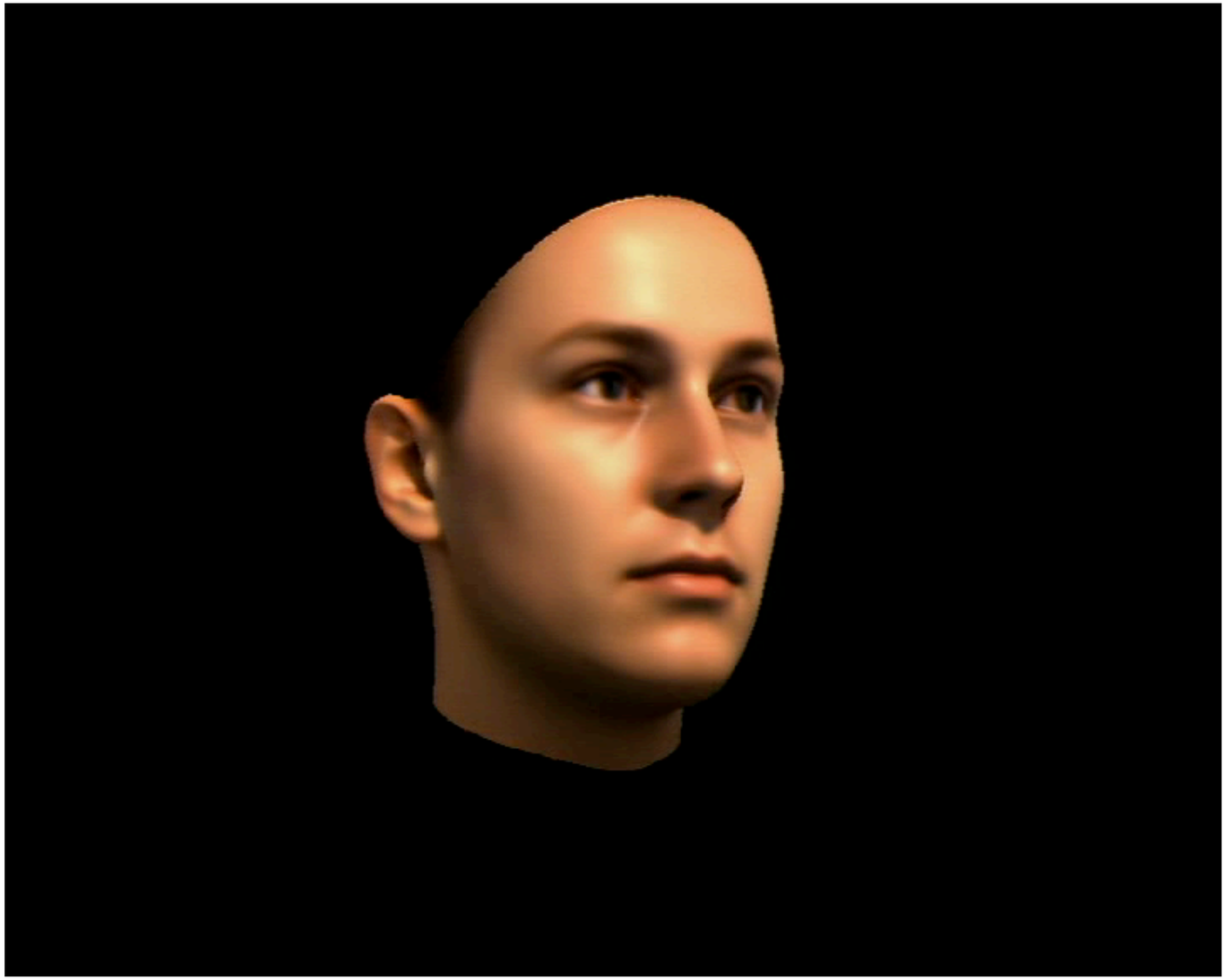

```
rng = np.random.RandomState(10)
X = np.dot(rng.rand(2, 2), rng.randn(2, 500)).T

mean_vec = np.mean(X, axis=0)
cov_mat = (X - mean_vec).T.dot((X - mean_vec)) / (X.shape[0]-1)
eig_vals, eig_vecs = np.linalg.eig(cov_mat)
```

Morphable Faces



Morphable Faces



Singular Value Decomposition (SVD)

- Very useful for matrix manipulation.
- Used for robust numerical computation.

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rotation

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scaling

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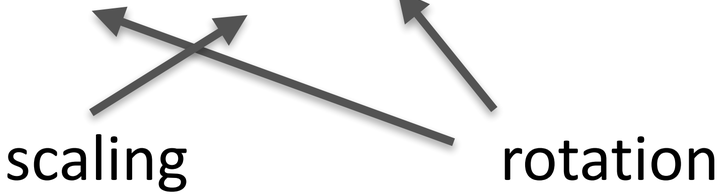
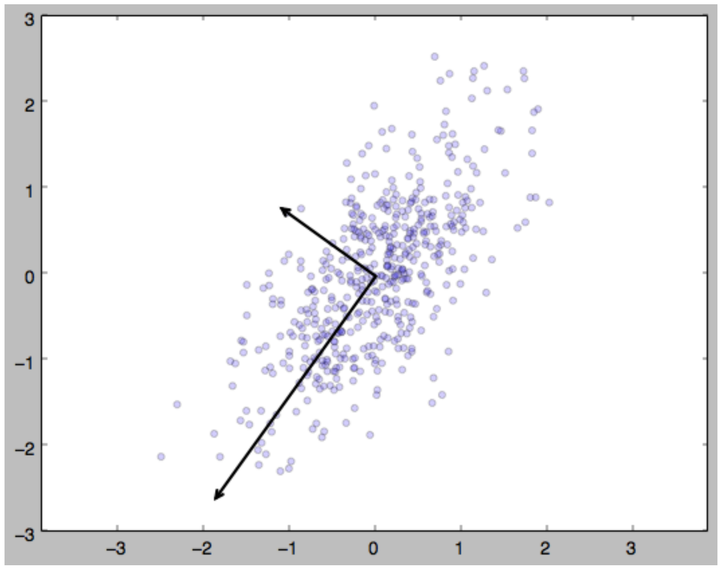


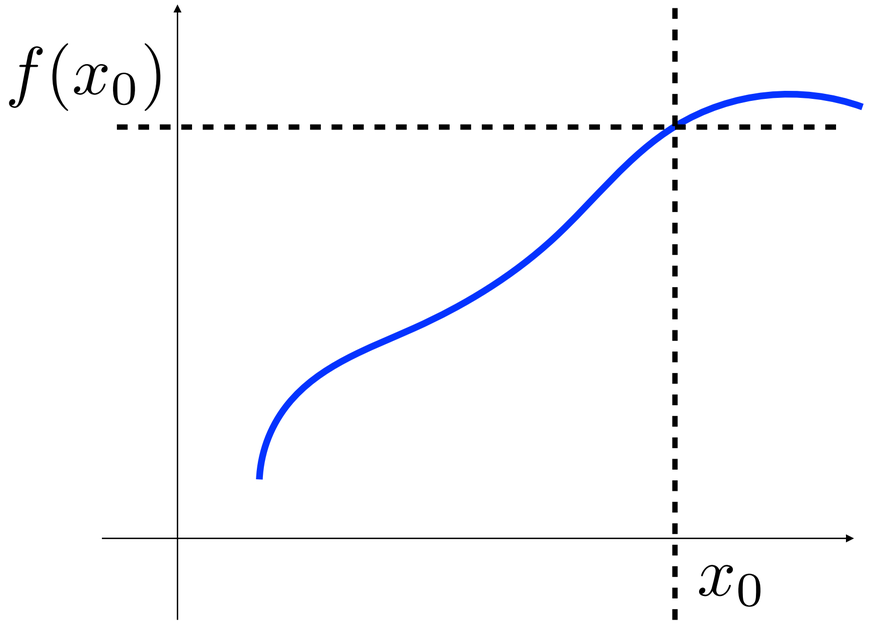
$$\mathbf{A} = \mathbf{A}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T$$

Code Example




```
mean_vec = np.mean(X, axis=0)
cov_mat = (X - mean_vec).T.dot((X - mean_vec)) / (X.shape[0]-1)
matU, sigma, matV = np.linalg.svd(cov_mat)
```

Differentiation (chain rule recap)



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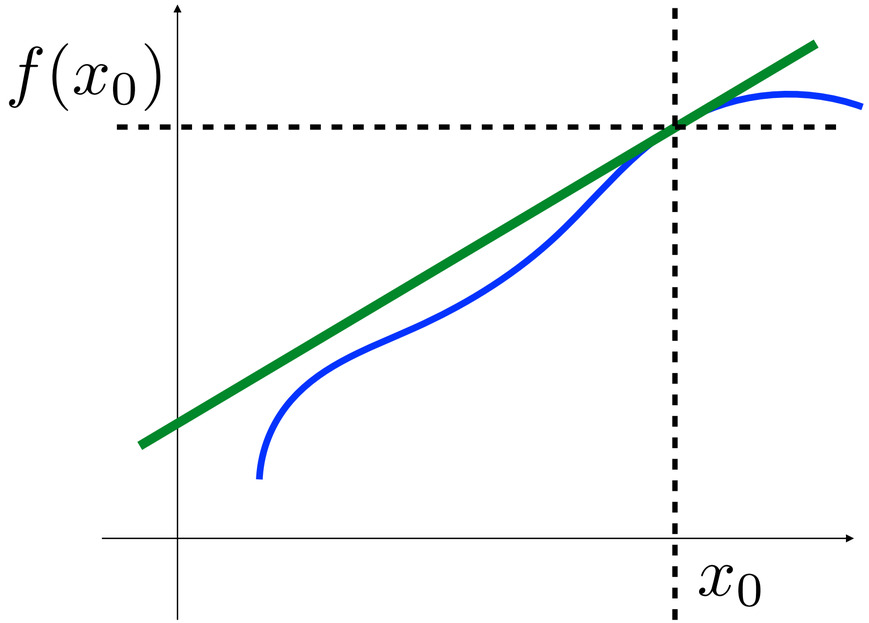

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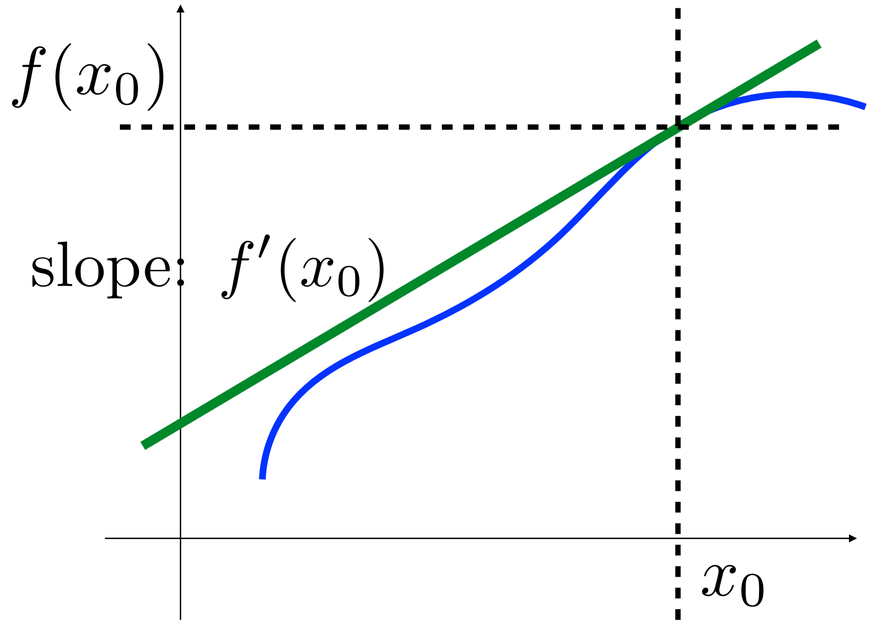
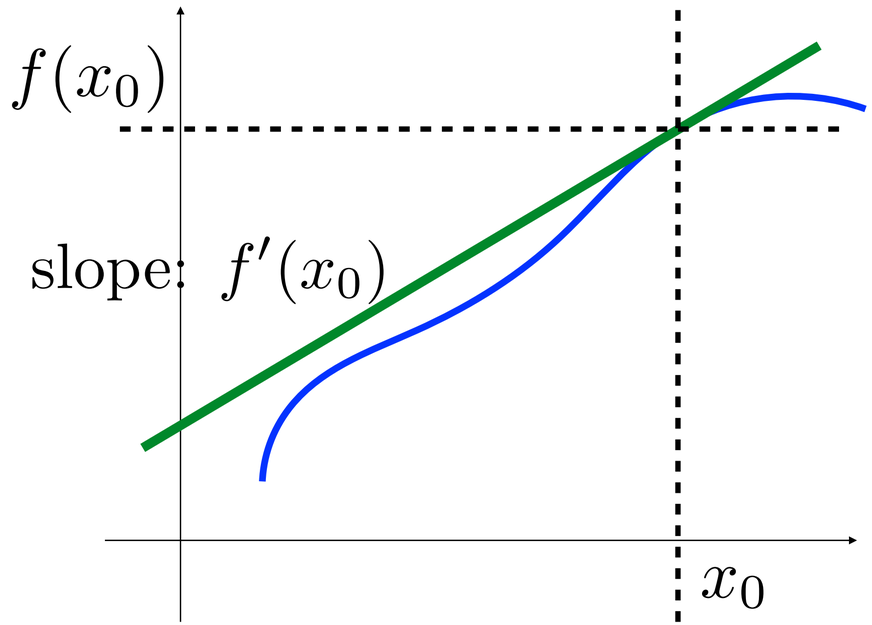


Differentiation (chain rule recap)

$$z = f \circ g(x) = f(g(x))$$



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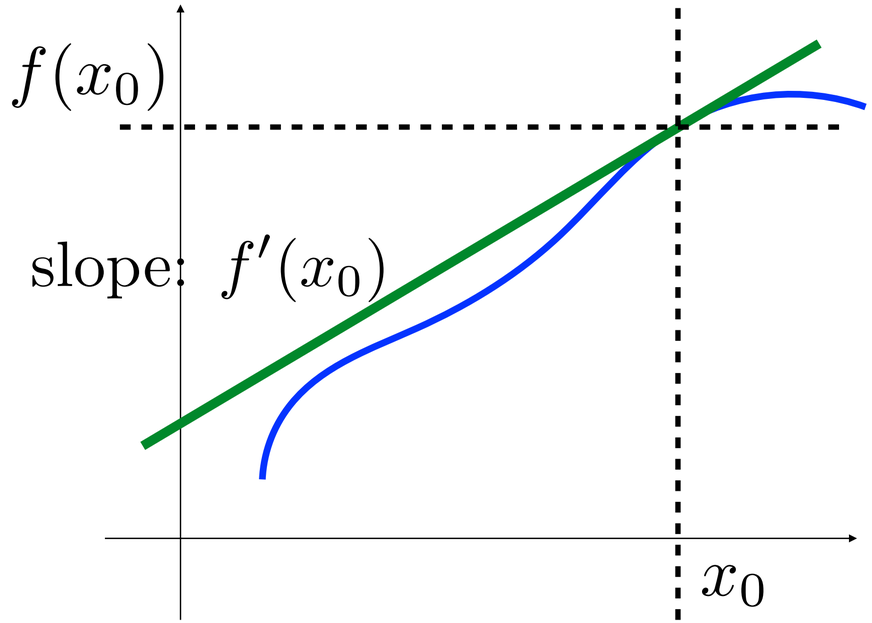

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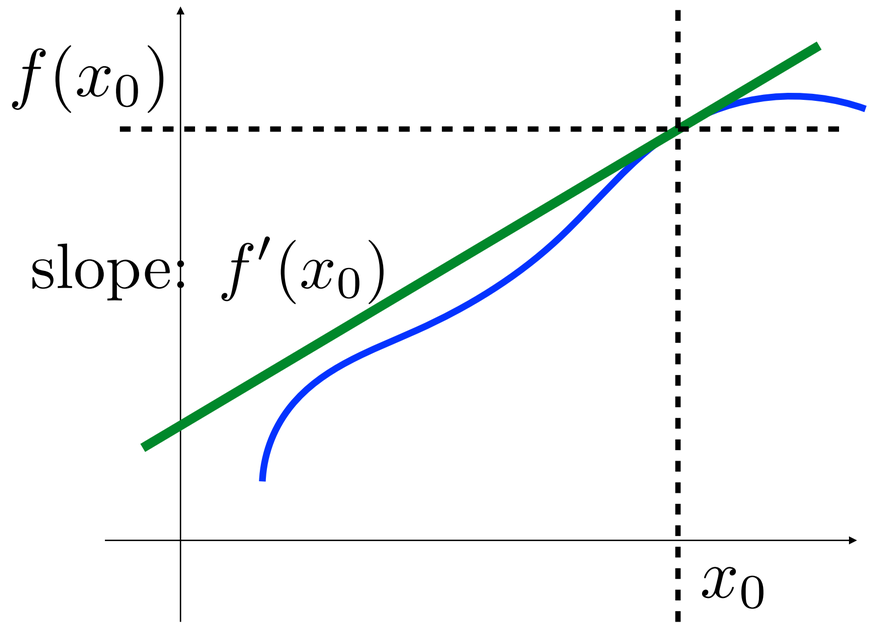

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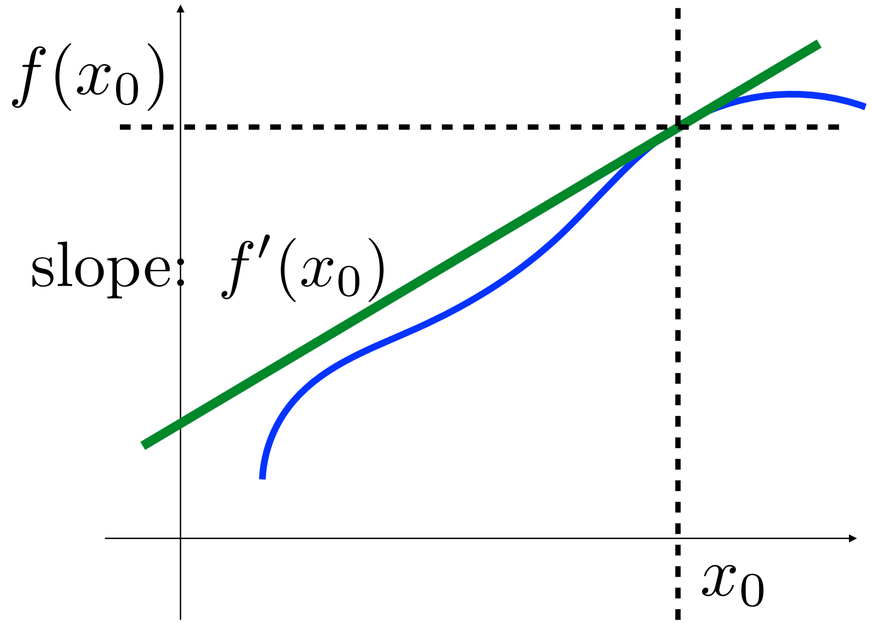

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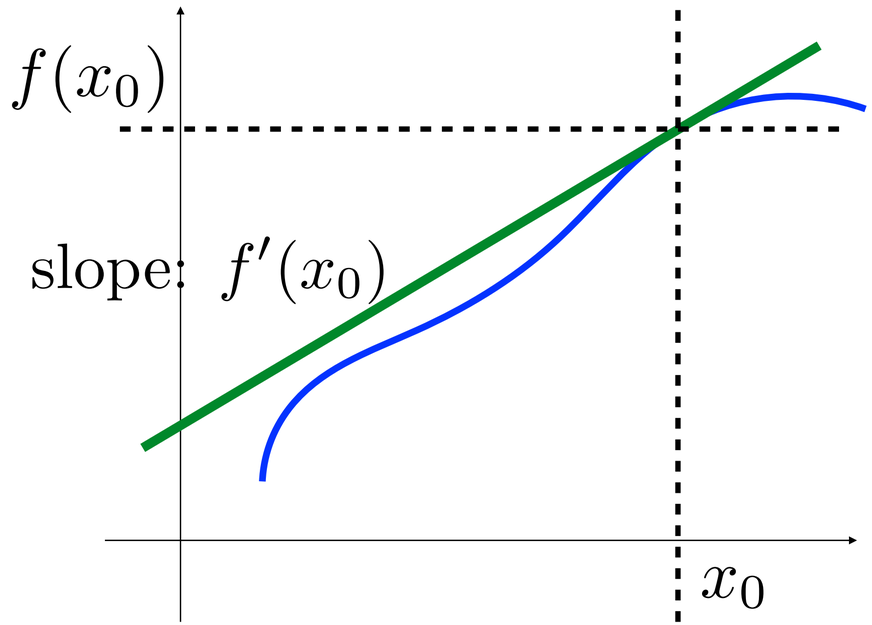

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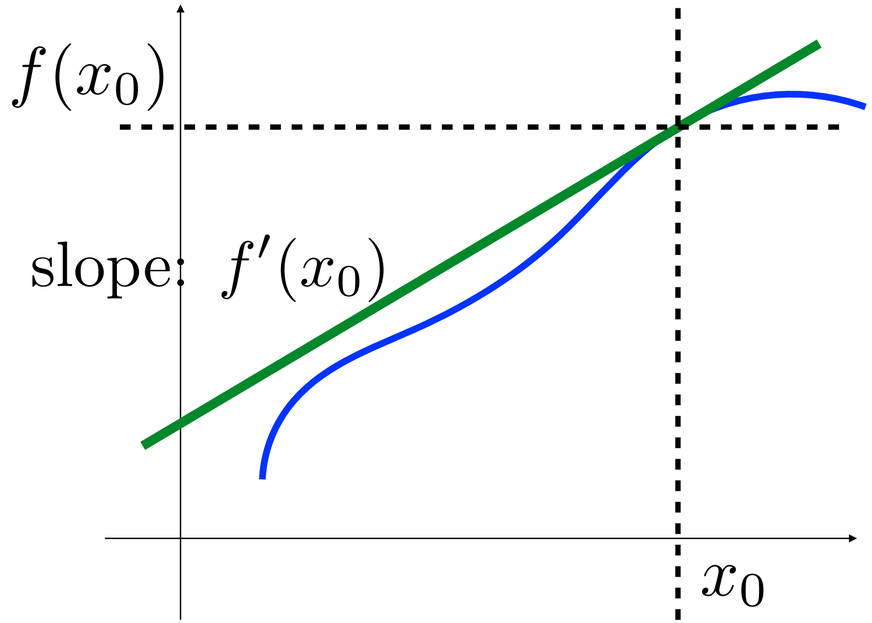
$z = \sin(5x)$



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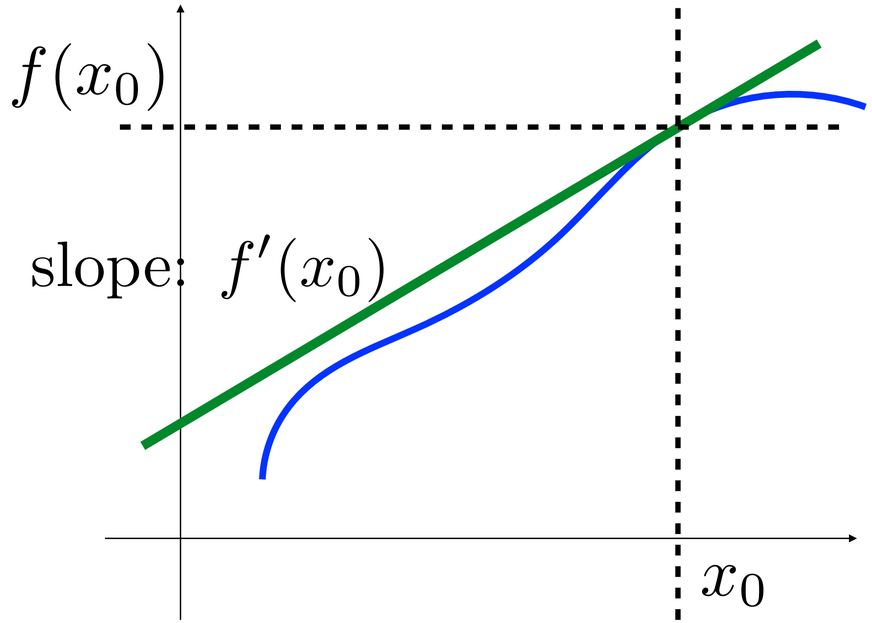
$$= \frac{d\sin(5x)}{d(5x)} \frac{d(5x)}{dx}$$



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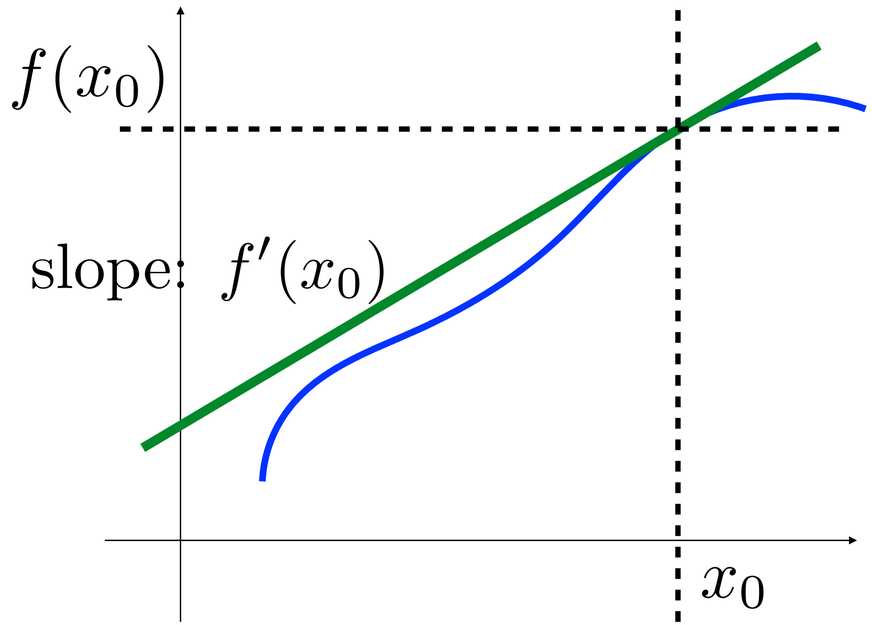
$$\begin{aligned} z &= \sin(5x) \\ &= \frac{d\sin(5x)}{d(5x)} \frac{d(5x)}{dx} \\ &= 5\cos(5x) \end{aligned}$$



Derivative Matrix

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

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Derivative Matrix

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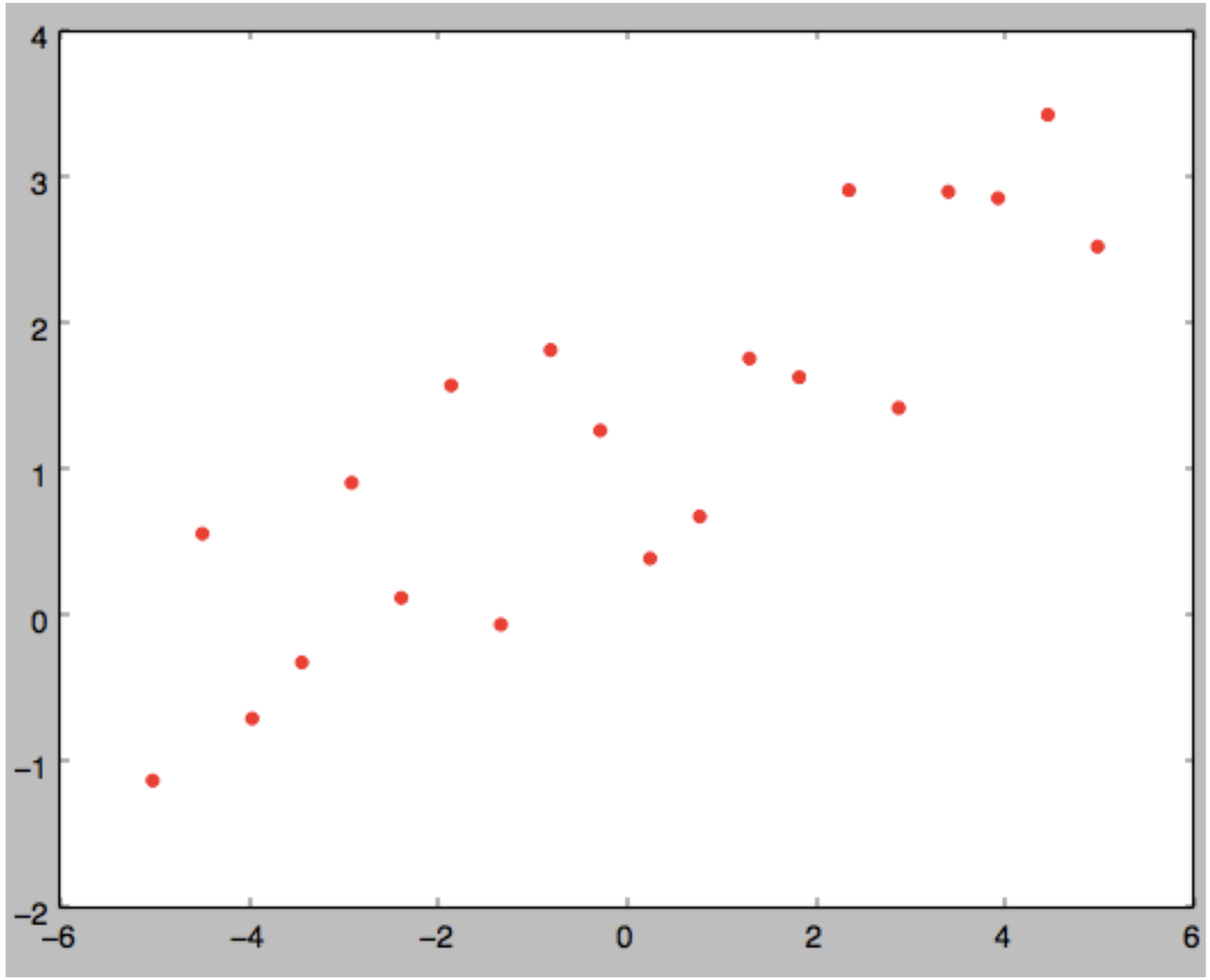
Jacobian matrix

$$\mathbf{L} = D\mathbf{f} = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix}_{m \times n}$$

Regression: Continuous Output



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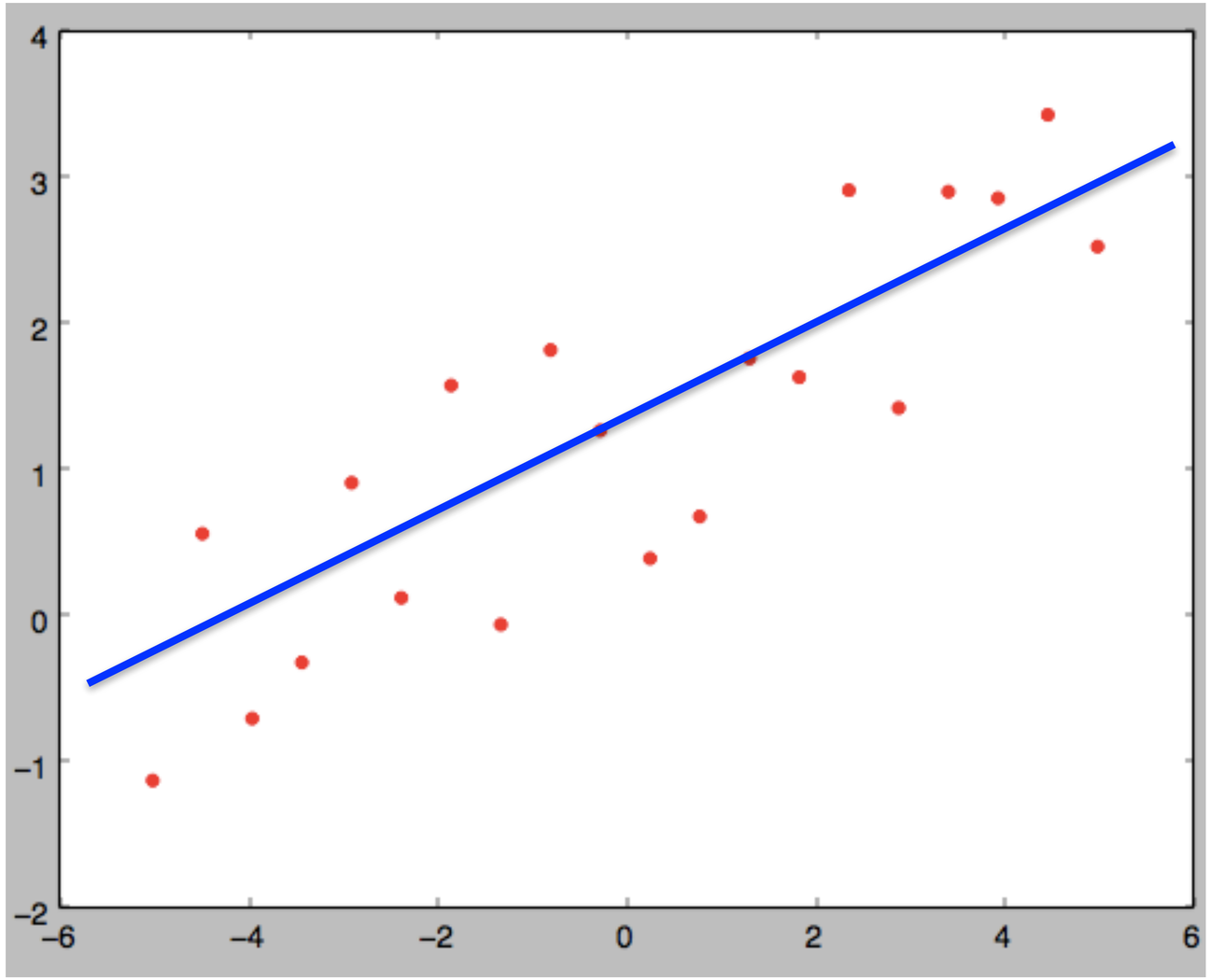
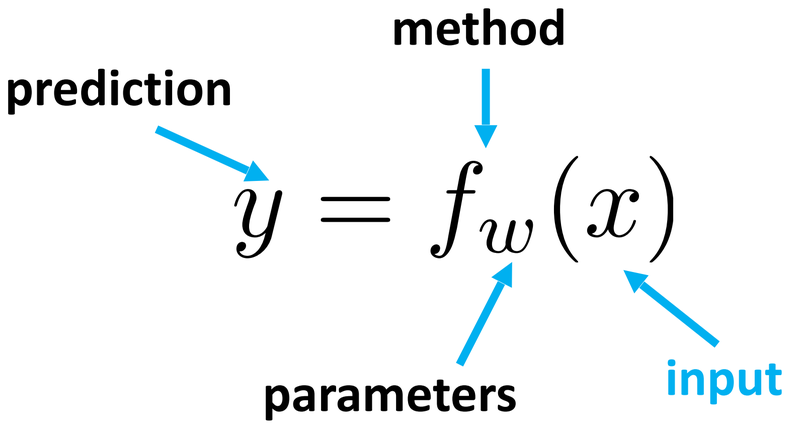


Learning a Function

$$y = f_w(x)$$

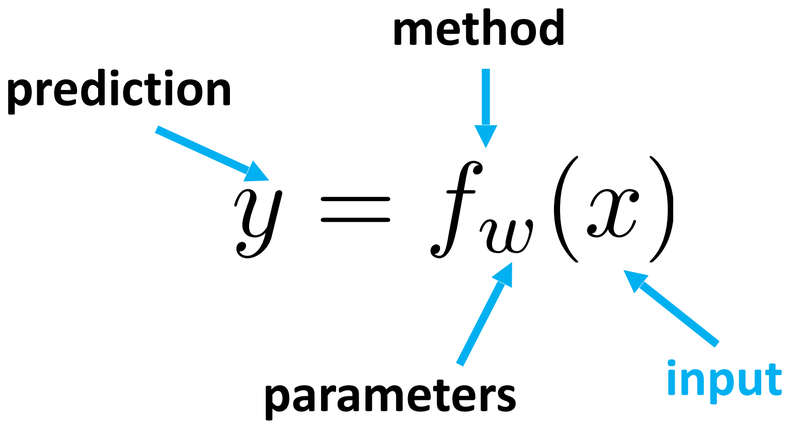
Learning a Function



Calculus $x \in \mathbb{R}$

Vector calculus $\mathbf{x} \in \mathbb{R}^d$

Learning a Function



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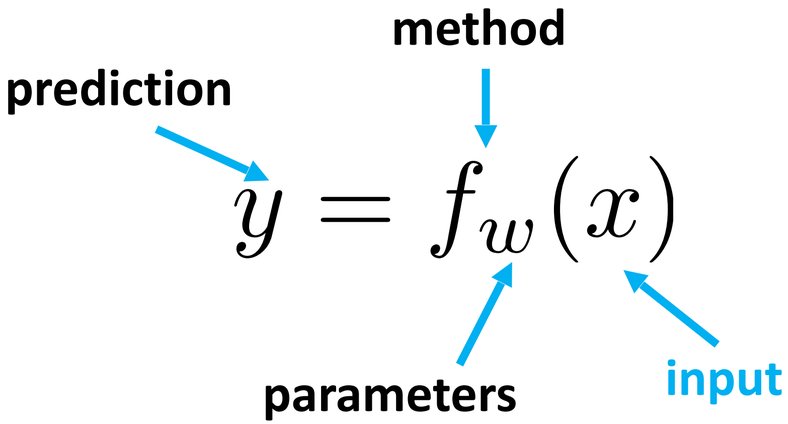
Machine learning: can work also for discrete inputs, strings, images, meshes, animations, …

Learning a Function



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Classification: $y \in \{0, 1\}$

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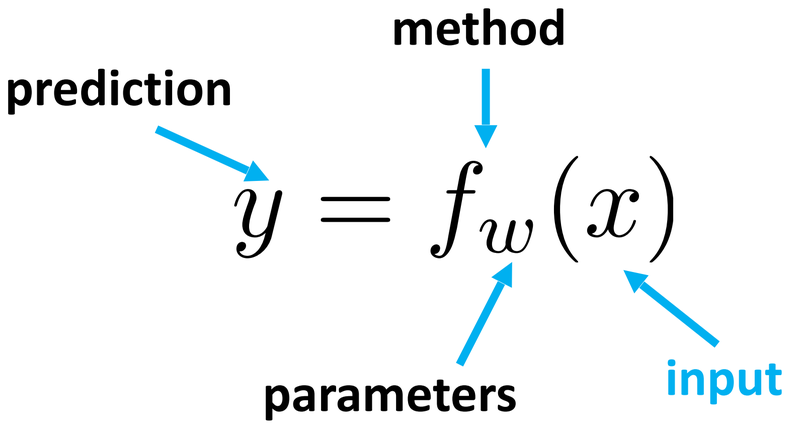
Learning a Function



Calculus $x \in \mathbb{R}$

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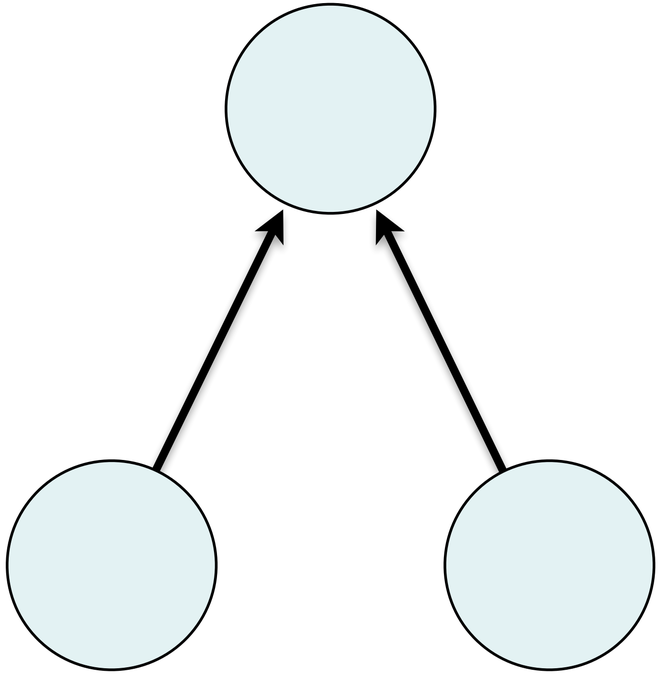
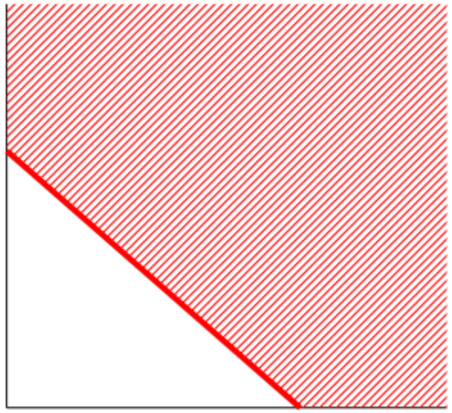
Classification: $y \in \{0, 1\}$

Regression: $y \in \mathbb{R}$

Machine learning: can work also for discrete inputs, strings, images, meshes, animations, …

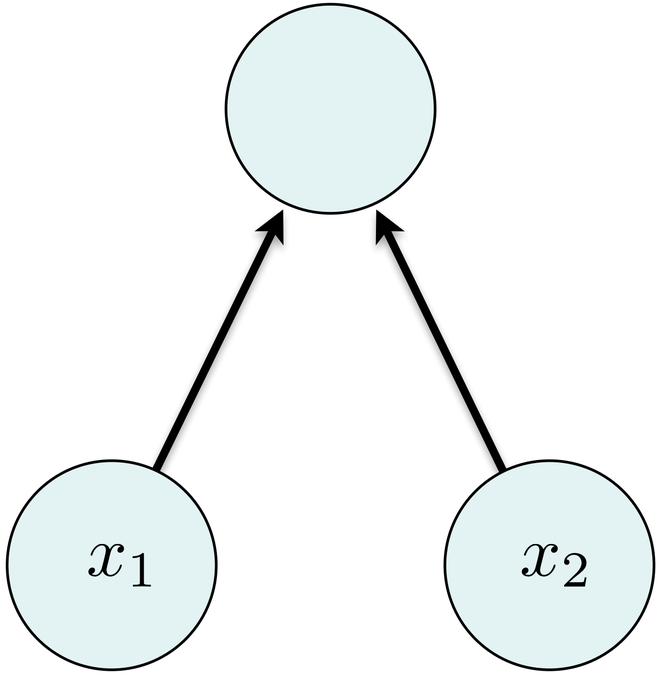
Learning a Simple Separator/Classifier

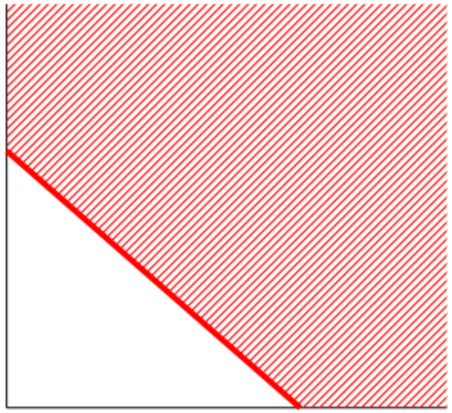




separating hyperplane

Learning a Simple Separator/Classifier





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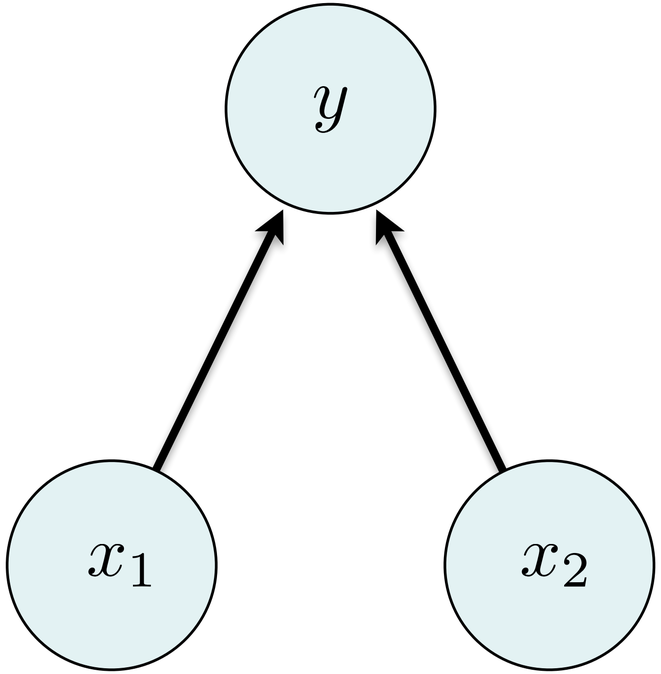

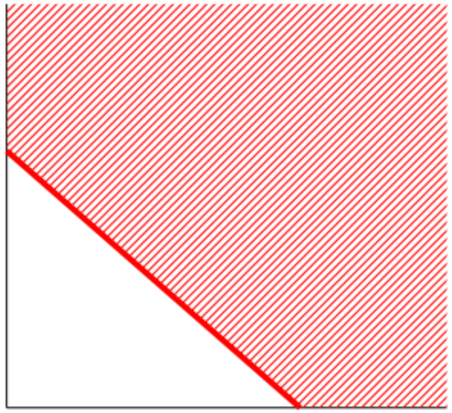

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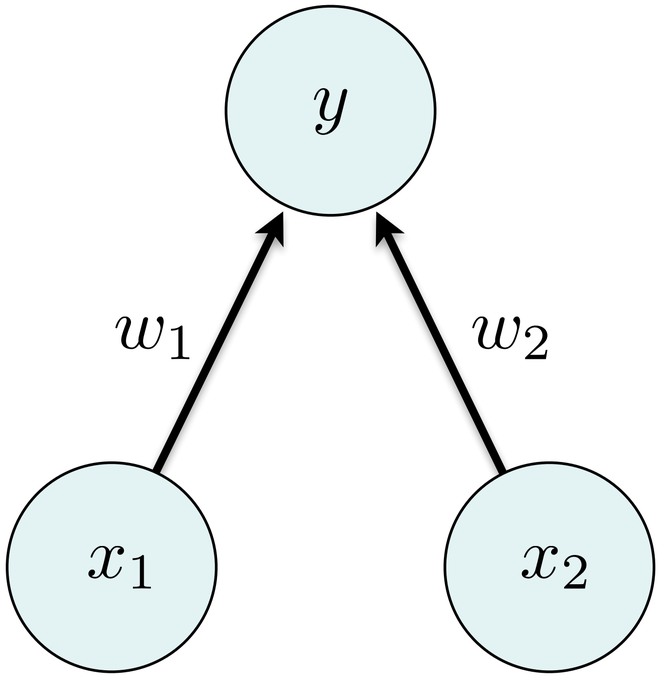

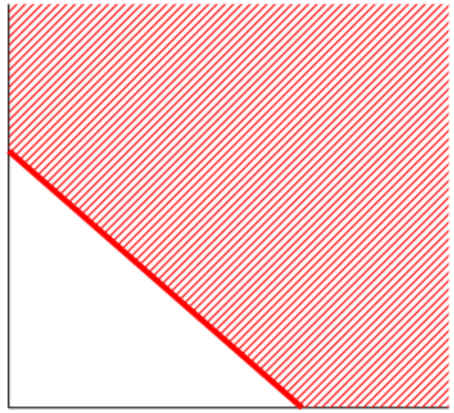

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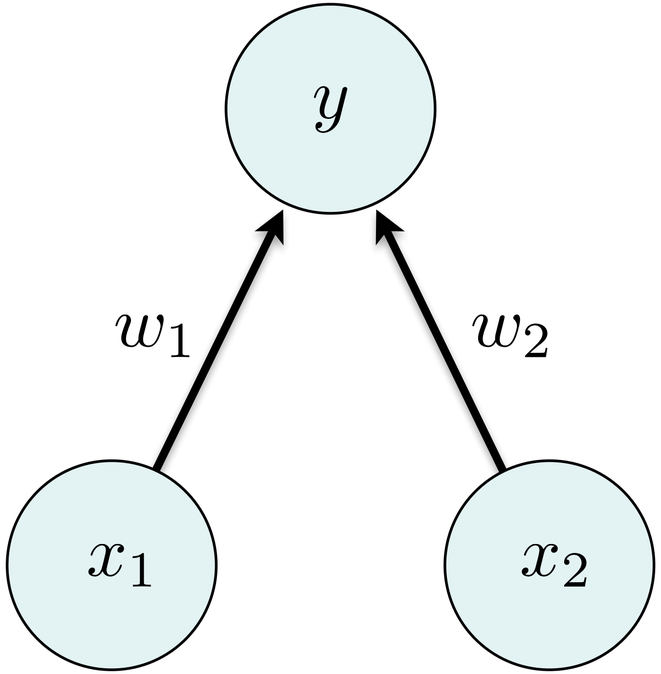
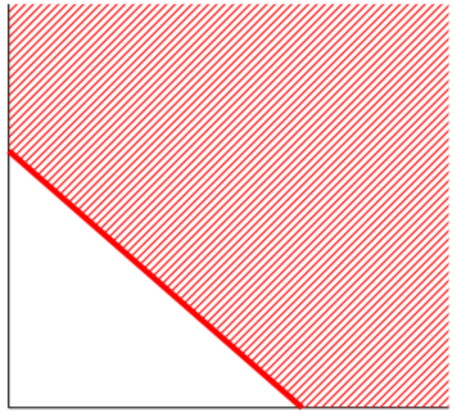
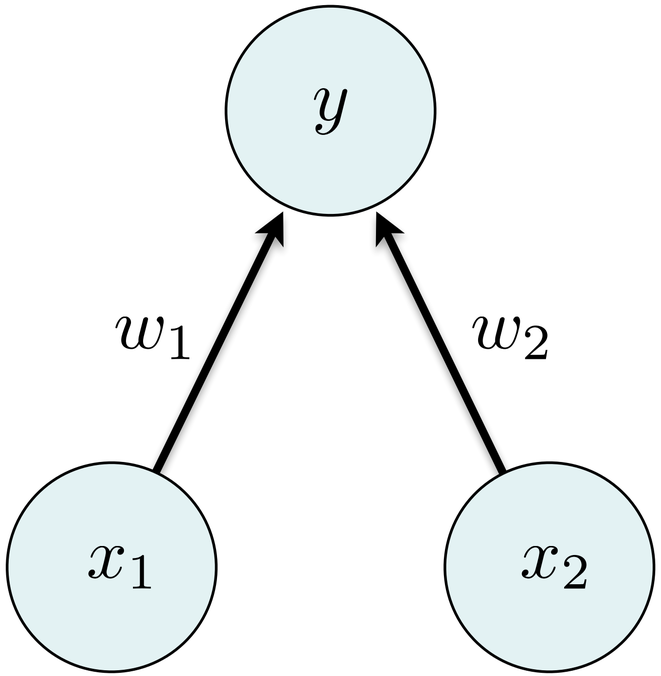


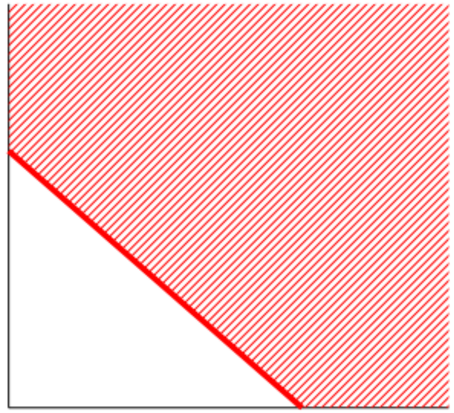


separating hyperplane

$$y = f(w_1 x_1 + w_2 x_2)$$

Learning a Simple Separator/Classifier

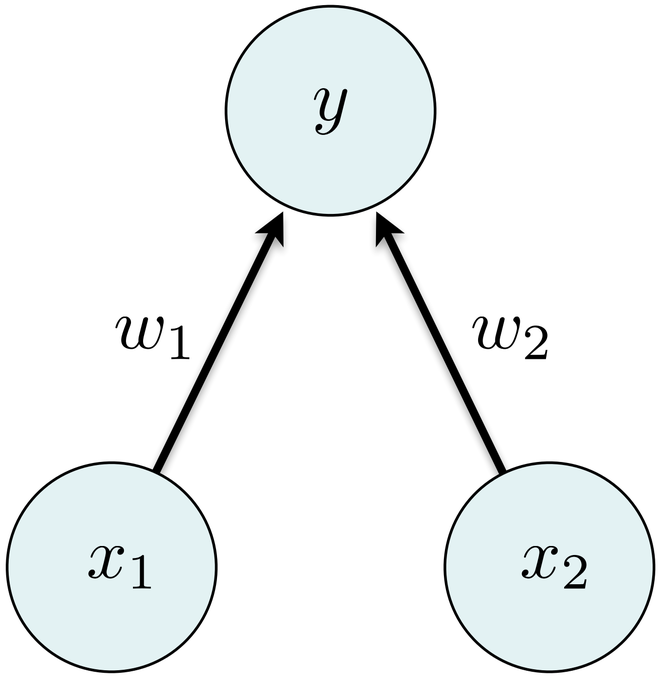


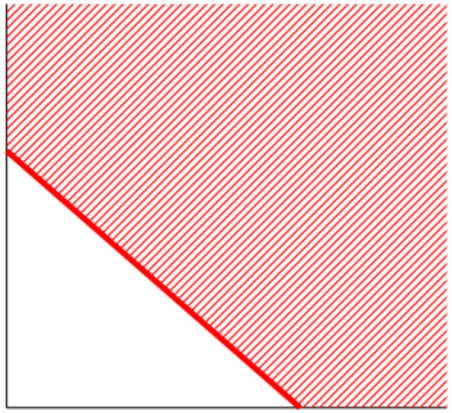


separating hyperplane

$$y = f(w_1 x_1 + w_2 x_2) = \mathcal{H}(w_1 x_1 + w_2 x_2)$$

Learning a Simple Separator/Classifier



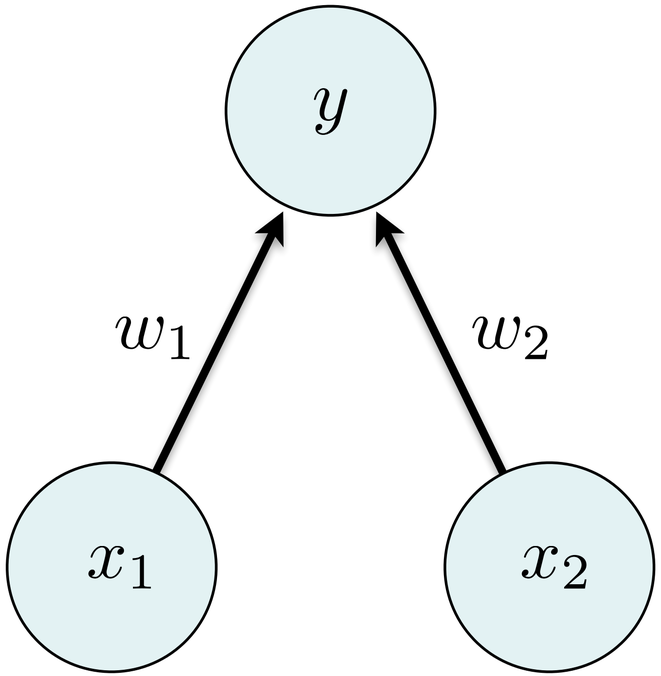


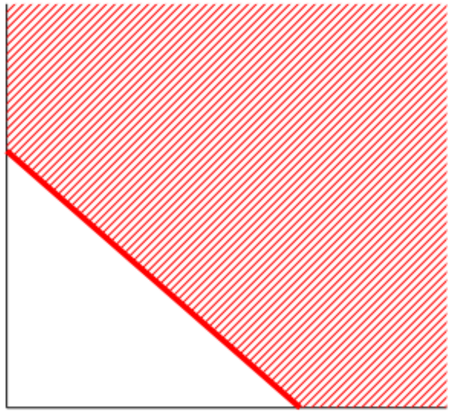
separating hyperplane

fixed non-linearity

$$y = f(w_1 x_1 + w_2 x_2) = \mathcal{H}(w_1 x_1 + w_2 x_2)$$

Learning a Simple Separator/Classifier





separating hyperplane

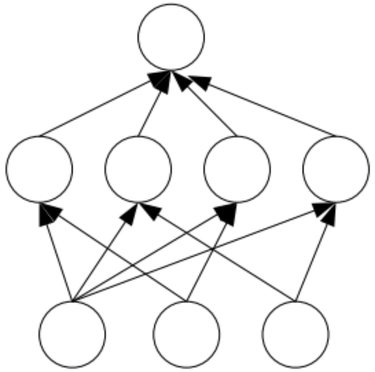
fixed non-linearity

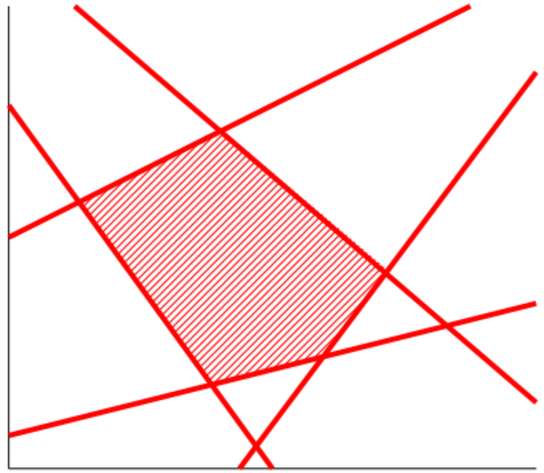
learned

$$y = f(w_1 x_1 + w_2 x_2) = \mathcal{H}(w_1 x_1 + w_2 x_2)$$

Combining Simple Functions/Classifiers

2 layers of trainable weights

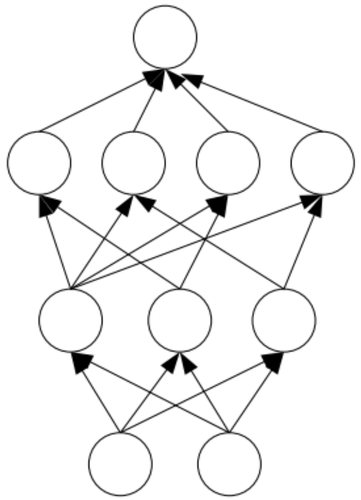


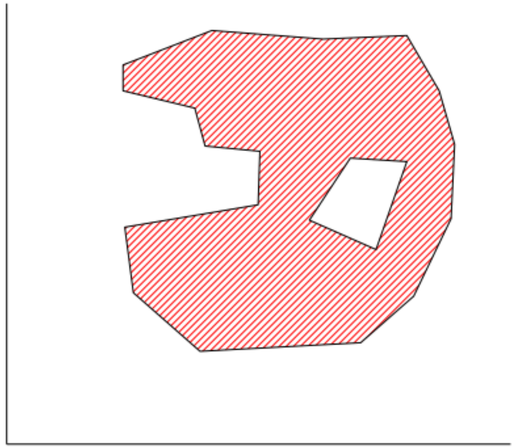


convex region

Combining Simple Functions/Classifiers

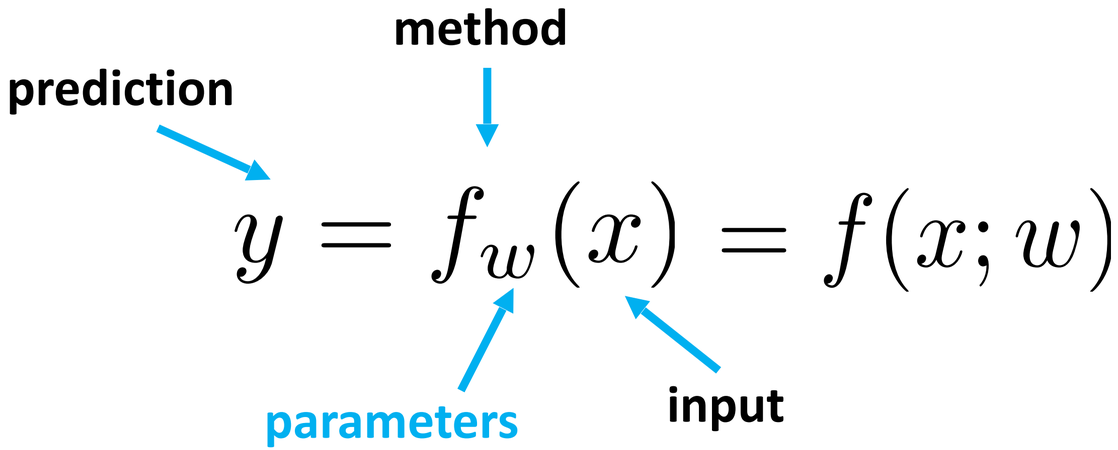
3 layers of trainable weights





complex polygons

Learning a Function: Modeling



$$w \in \mathbb{R}$$

$$\mathbf{w} \in \mathbb{R}^K$$

Regression

1. Least Squares fitting

2. Nonlinear error function and gradient descent

3. Perceptron training

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Assumption: Linear Function

$$y = f_{\mathbf{w}}(\mathbf{x}) = f(\mathbf{x}, \mathbf{w}) = \mathbf{w}^T \mathbf{x}$$

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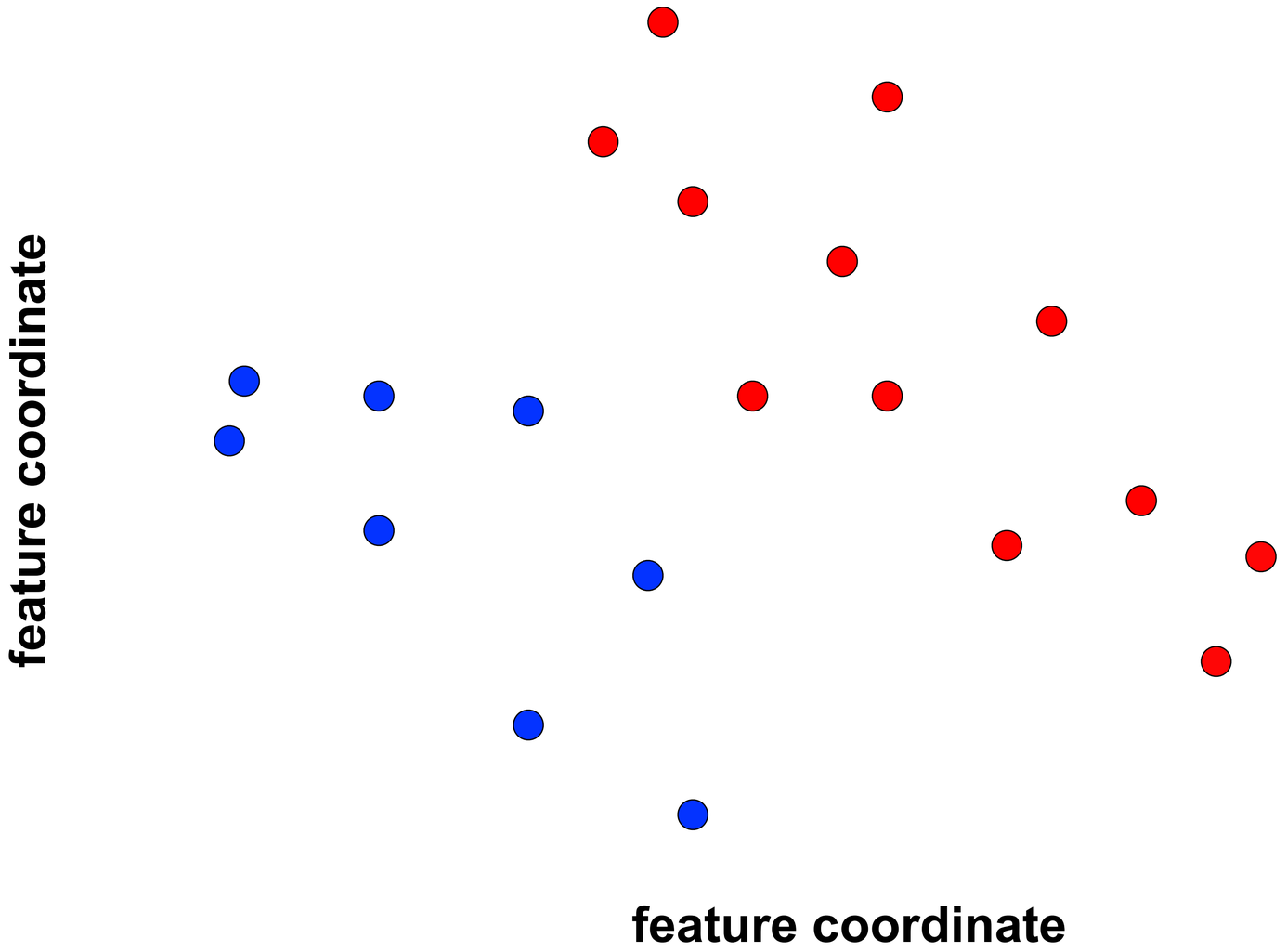
$$\mathbf{w}^T \mathbf{x} = \langle \mathbf{w}, \mathbf{x} \rangle = \sum_{d=1}^{D} \mathbf{w}_d \mathbf{x}_d$$

$$\mathbf{x} \in \mathbb{R}^D, \mathbf{w} \in \mathbb{R}^D$$

Reminder: Linear Classifier



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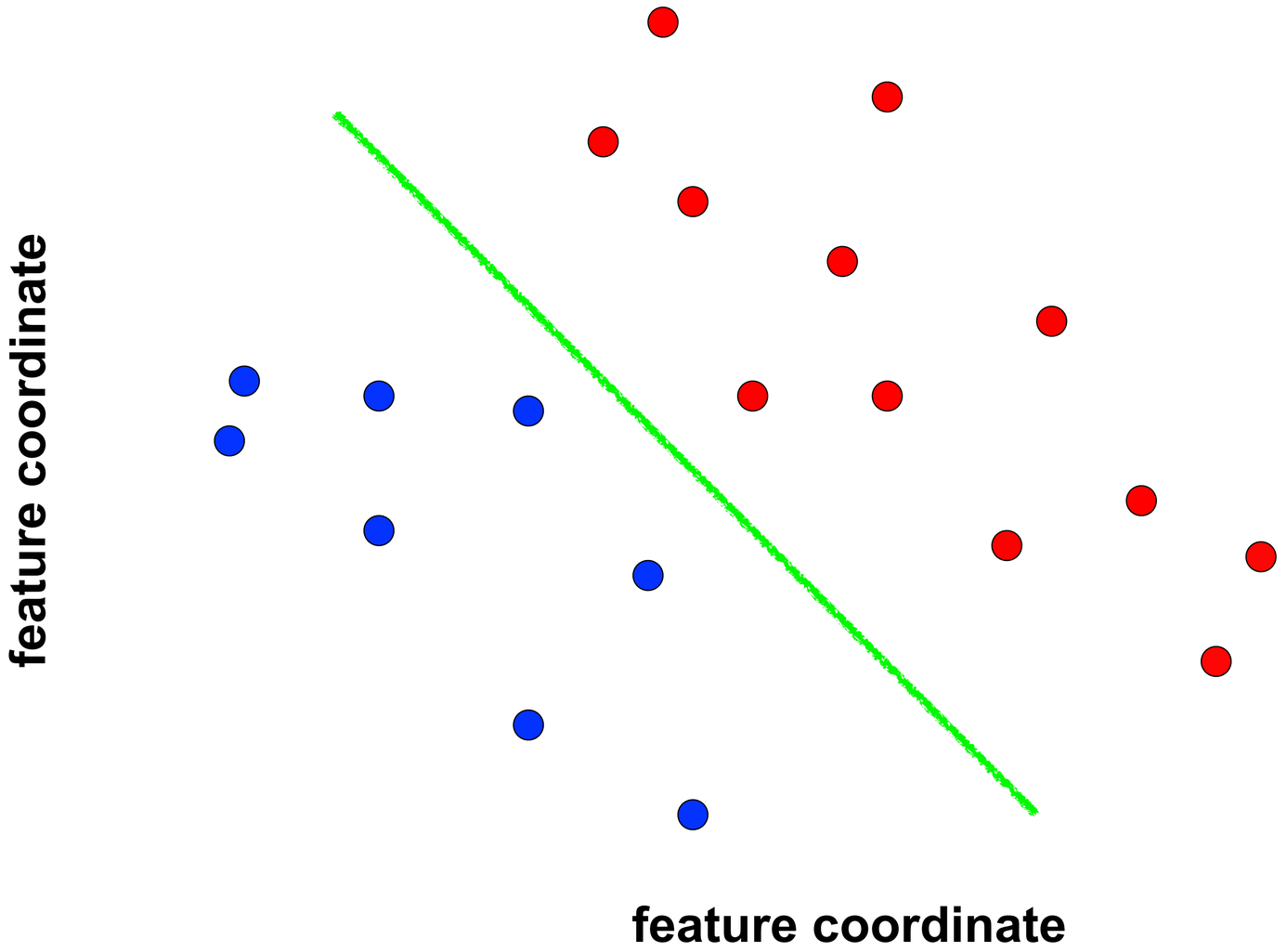

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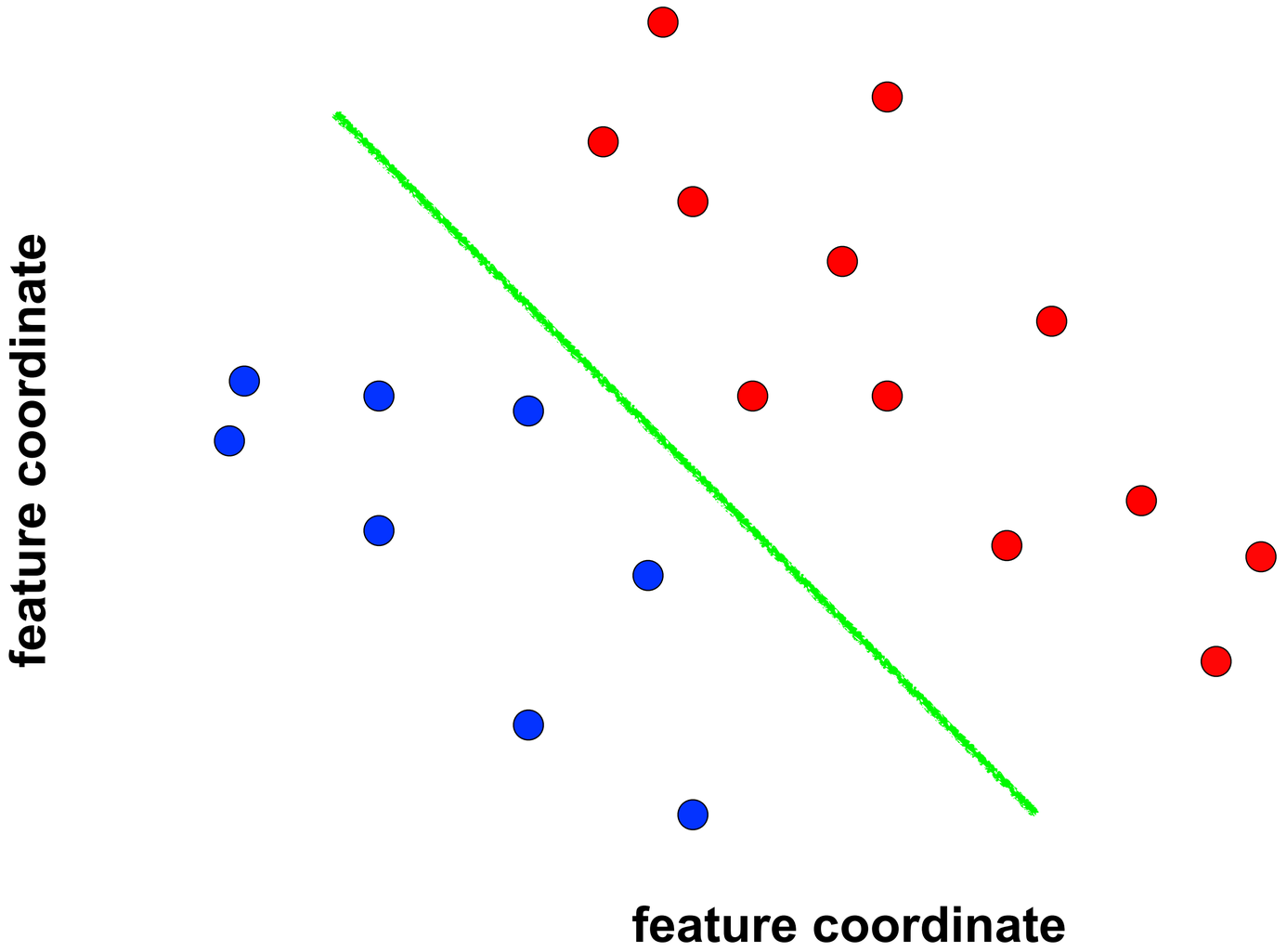

$\mathbf{x}_i$ positive: $\mathbf{x}_i \cdot \mathbf{w} \geq 0$

$\mathbf{x}_i$ negative: $\mathbf{x}_i \cdot \mathbf{w} < 0$

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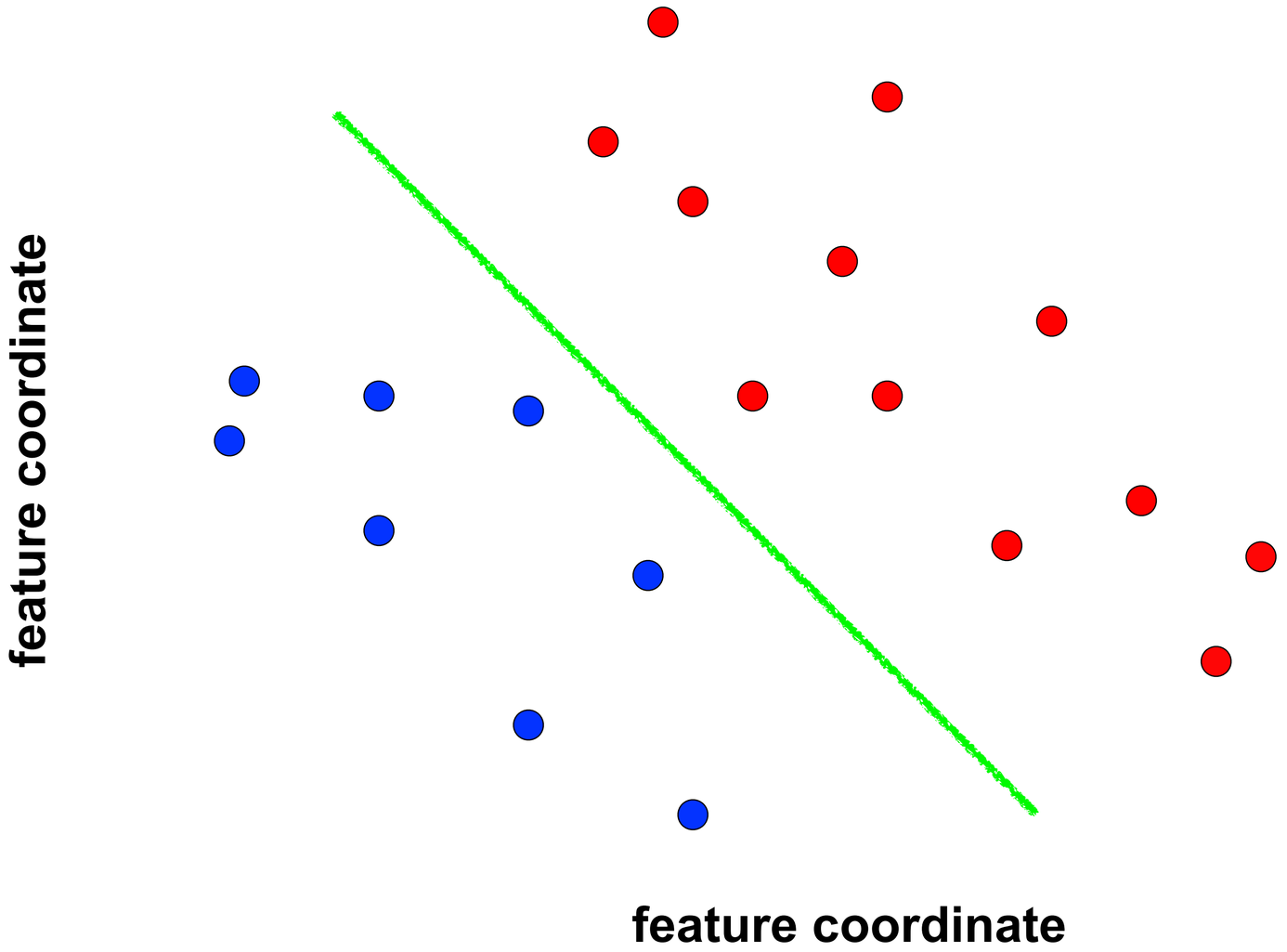

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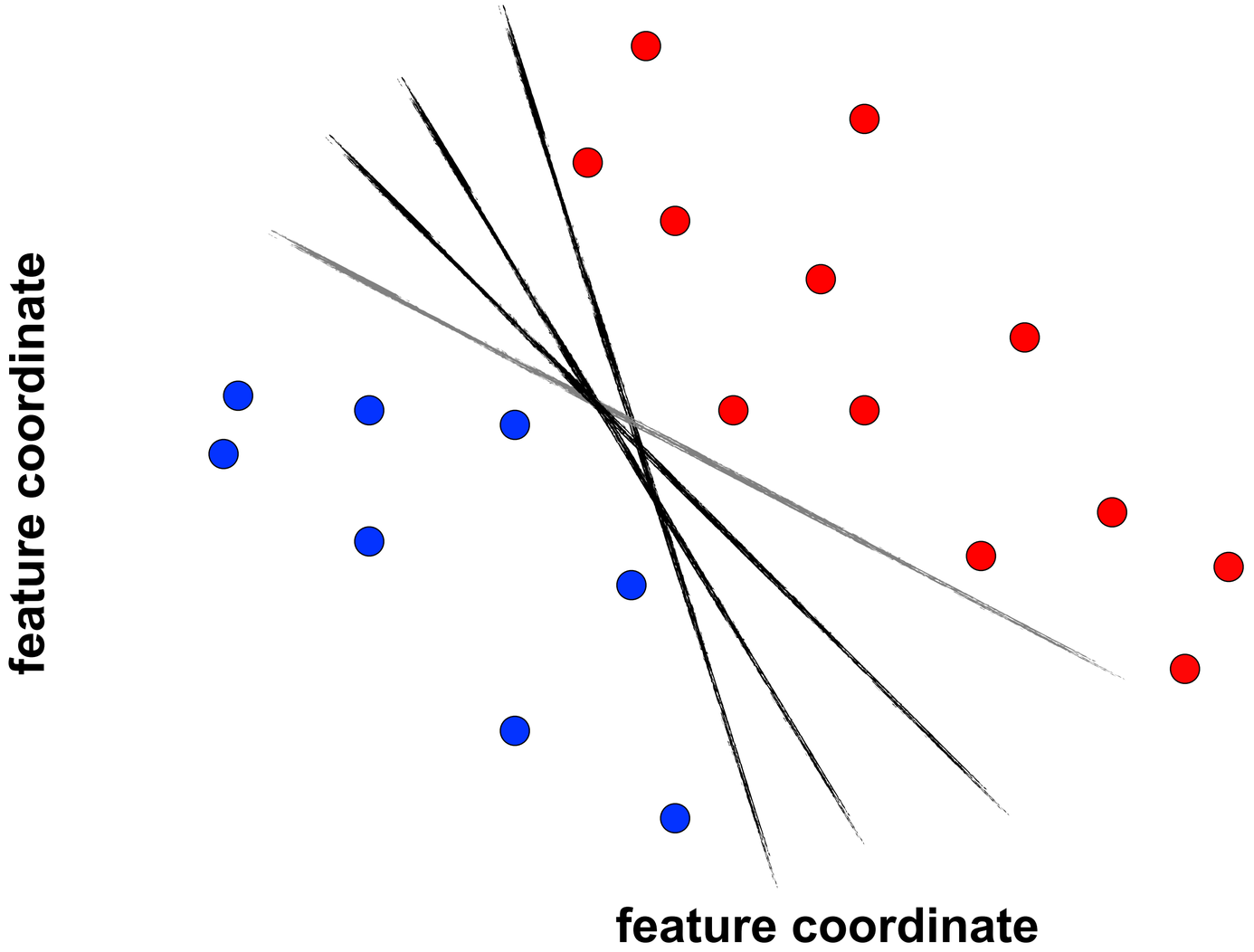
supervised setting

labelled input

$$y_t = \begin{cases} +1 & \bullet \\ -1 & \bullet \end{cases}$$

Which Line to Pick?



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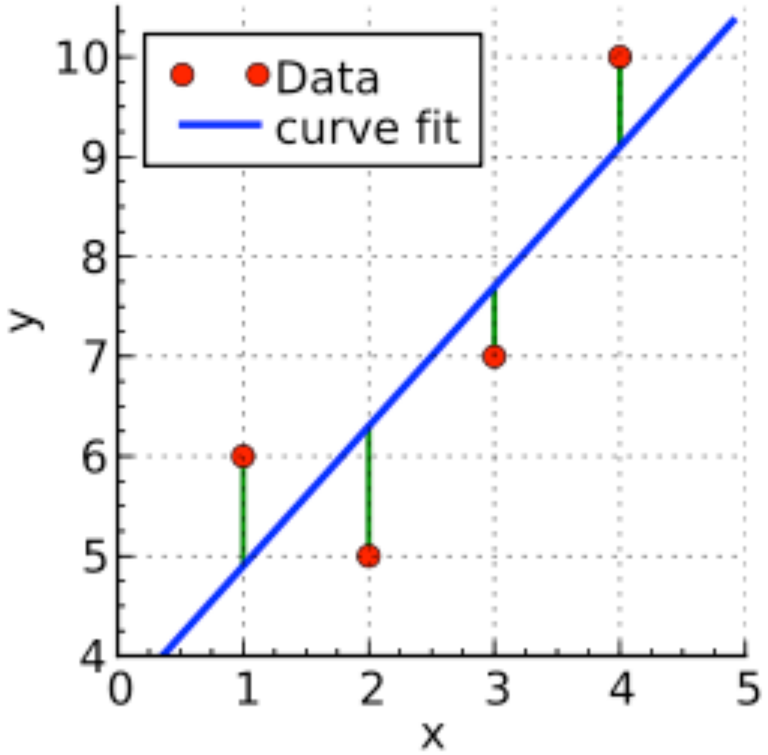
Linear Regression in 1D



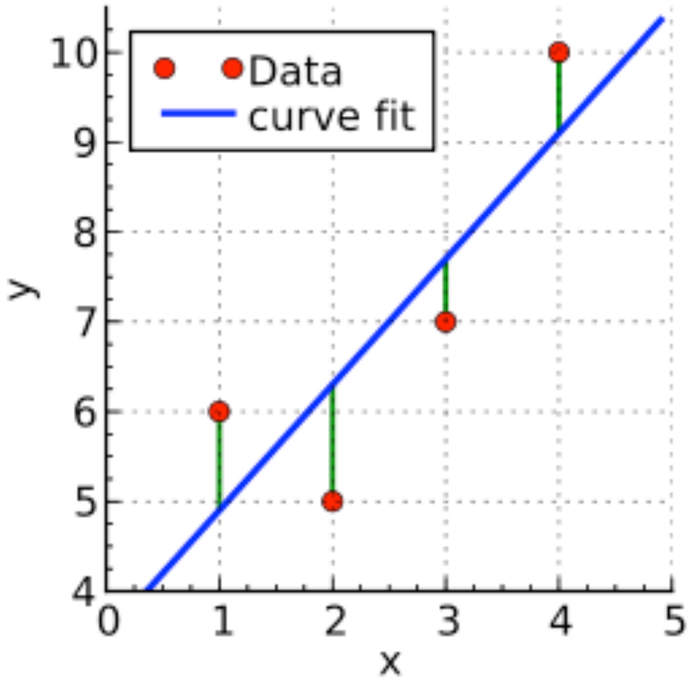
Training set: input–output pairs

$$\mathcal{S} = \{(x^i, y^i)\}, \quad i = 1 \ldots, N$$

$$x^i \in \mathbb{R}, \quad y^i \in \mathbb{R}$$

Linear regression in 1D

$$y^i = w_0 + w_1 x_1^i + \epsilon^i$$



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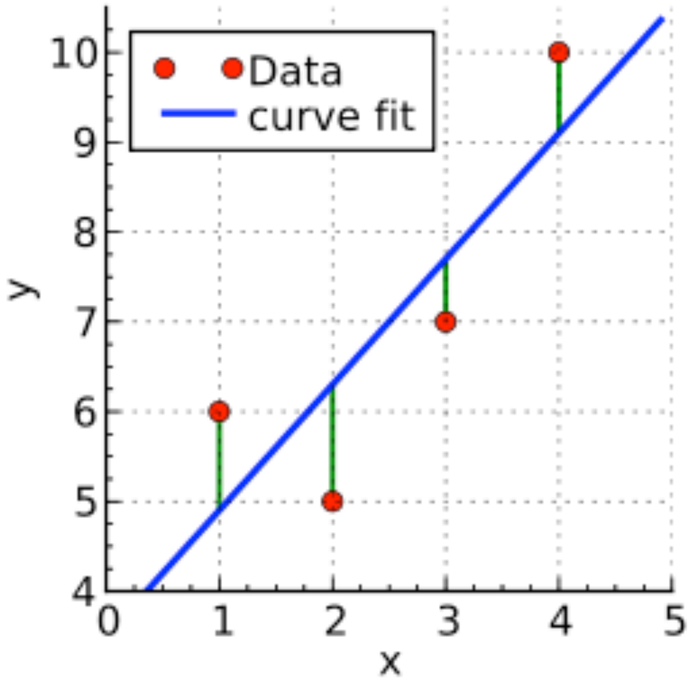
$$y^i = w_0 + w_1 x_1^i + \epsilon^i$$

$$= w_0 x_0^i + w_1 x_1^i + \epsilon^i, \quad x_0^i = 1, \quad \forall i$$



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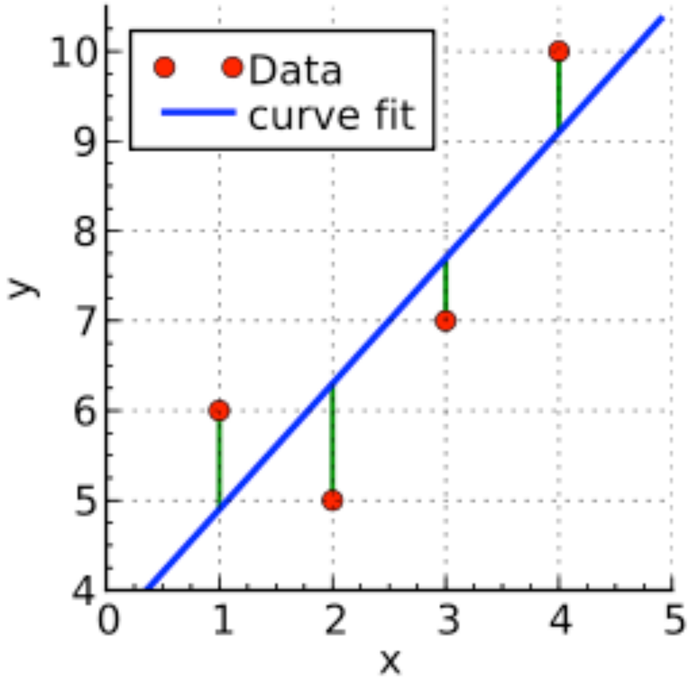

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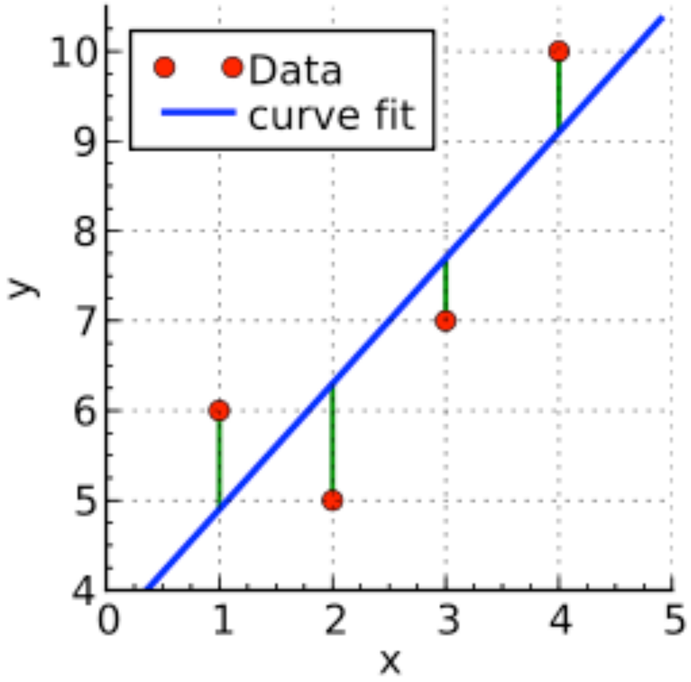

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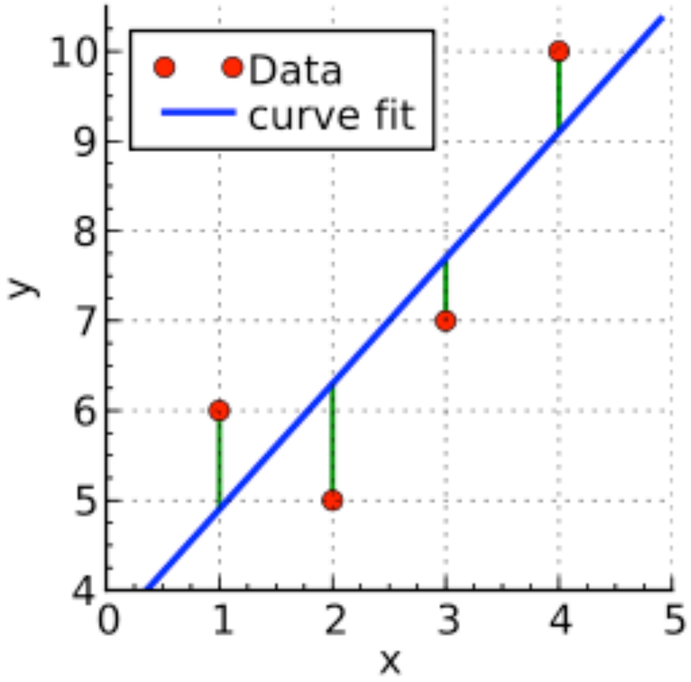
noise



Sum of Square Errors *(MSE without the mean)*

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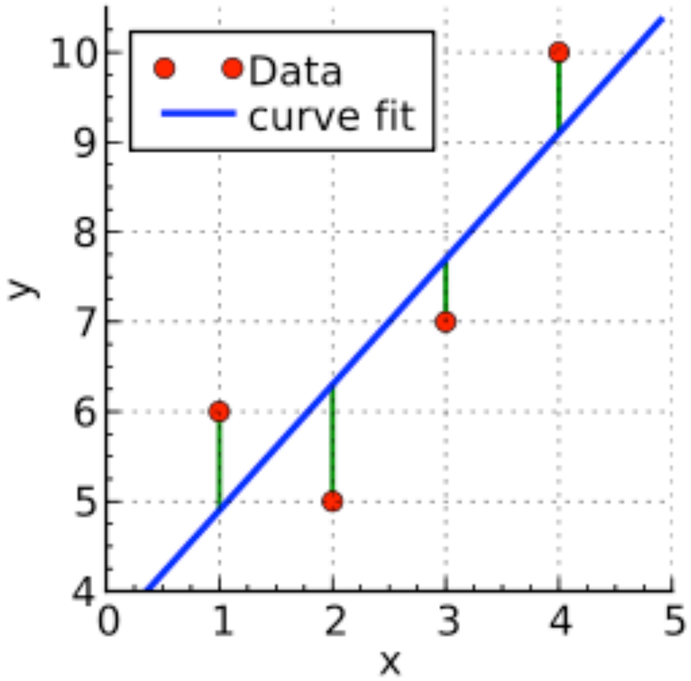

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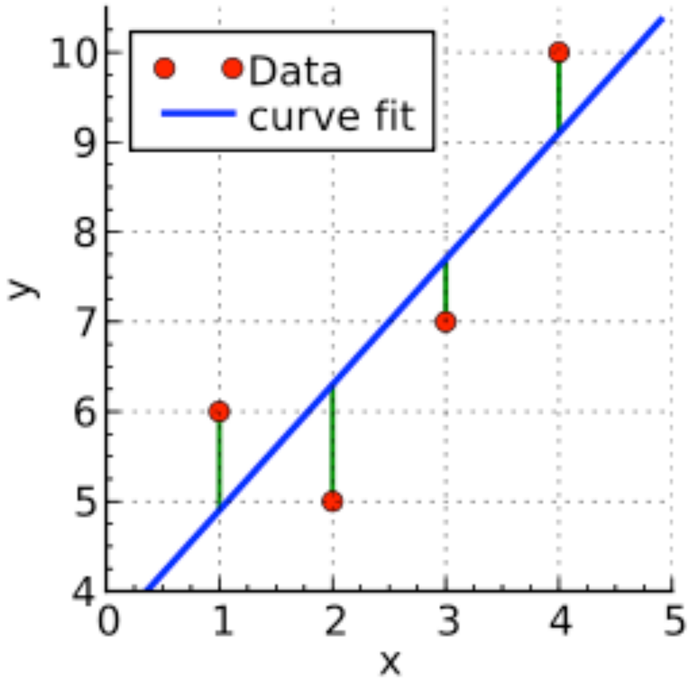

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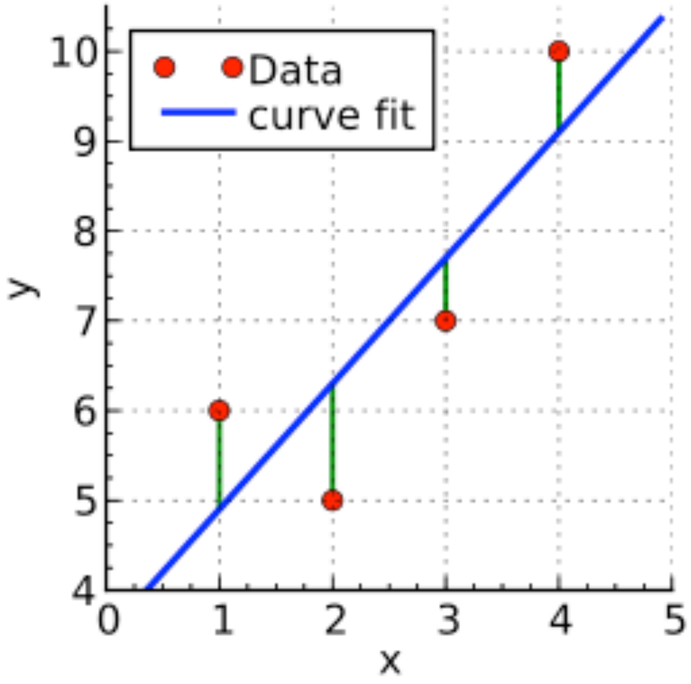
In two variables:

$$L(w_0, w_1) = \sum_{i=1}^{N} \left[y^i - \left(w_0 x_0^i + w_1 x_1^i \right) \right]^2$$



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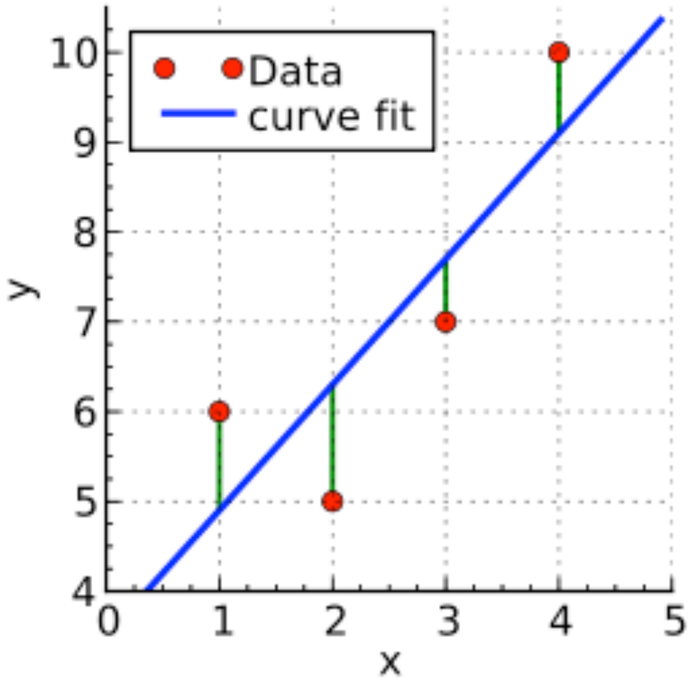
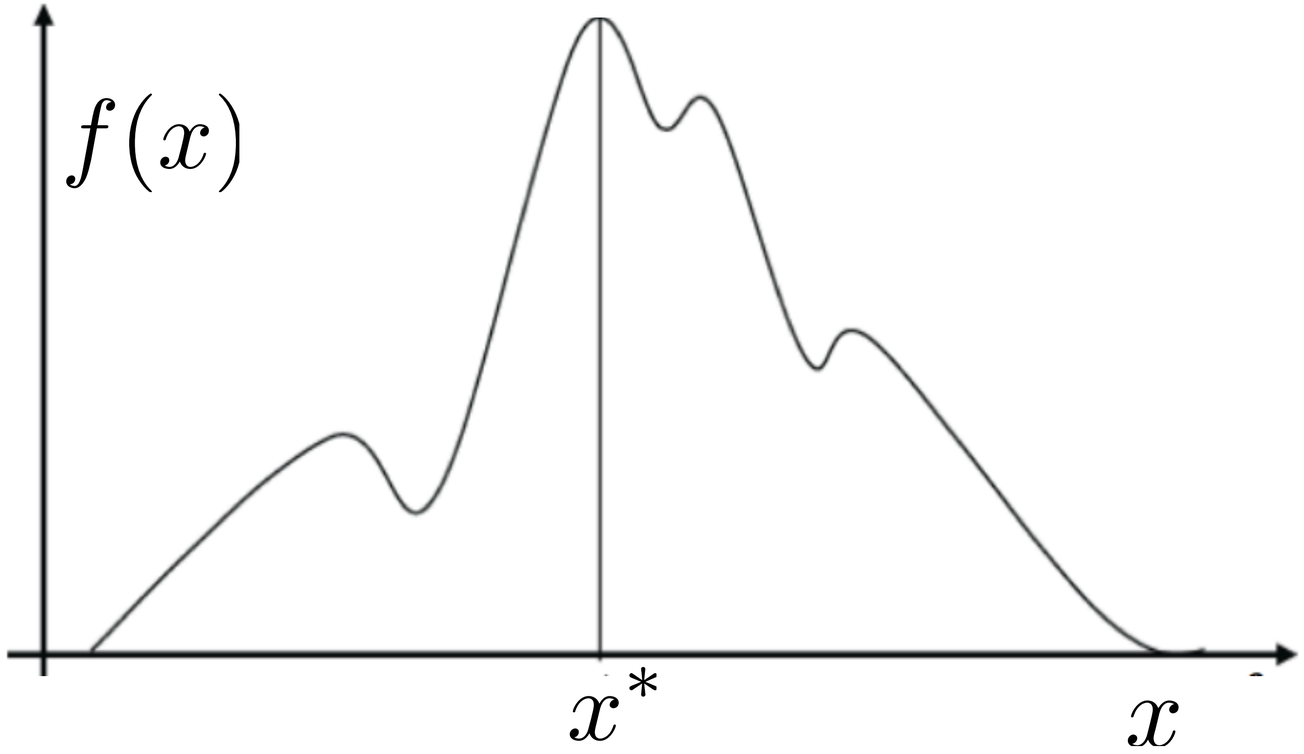
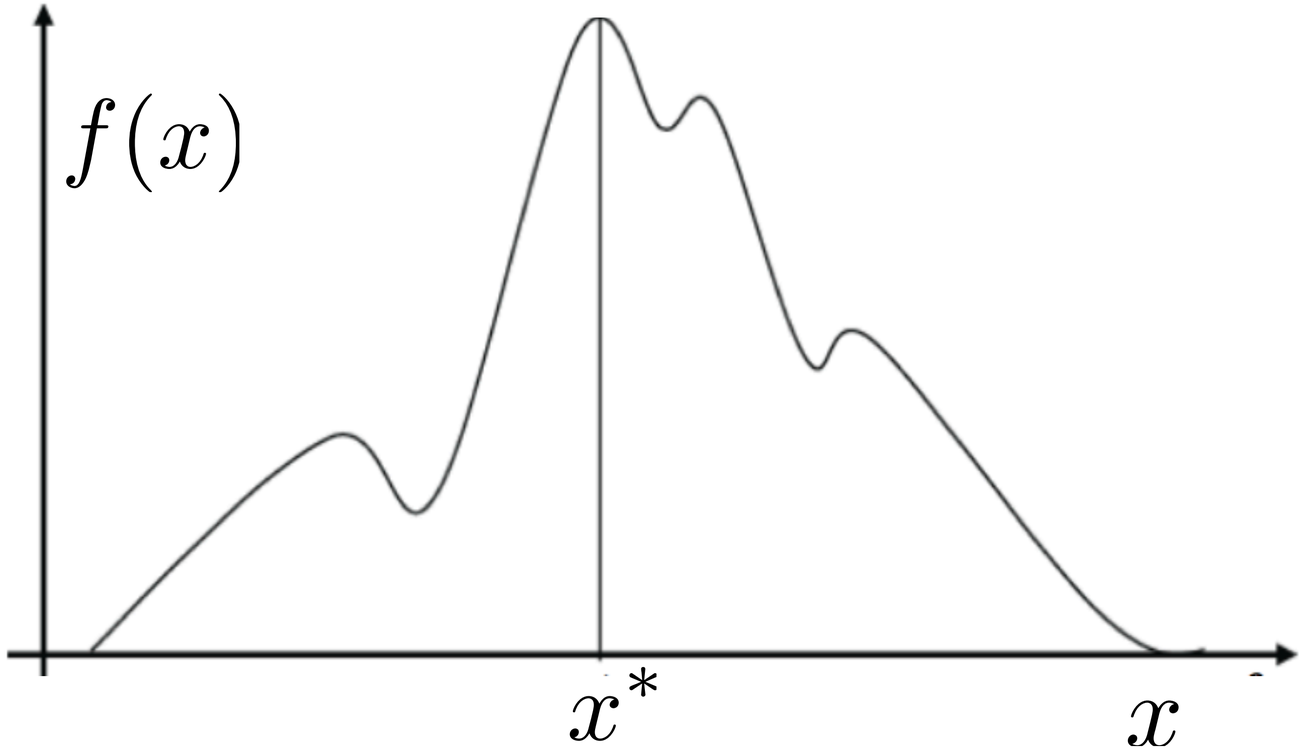


Question: what is the best (or least bad) value of ***w****?*

Calculus 101

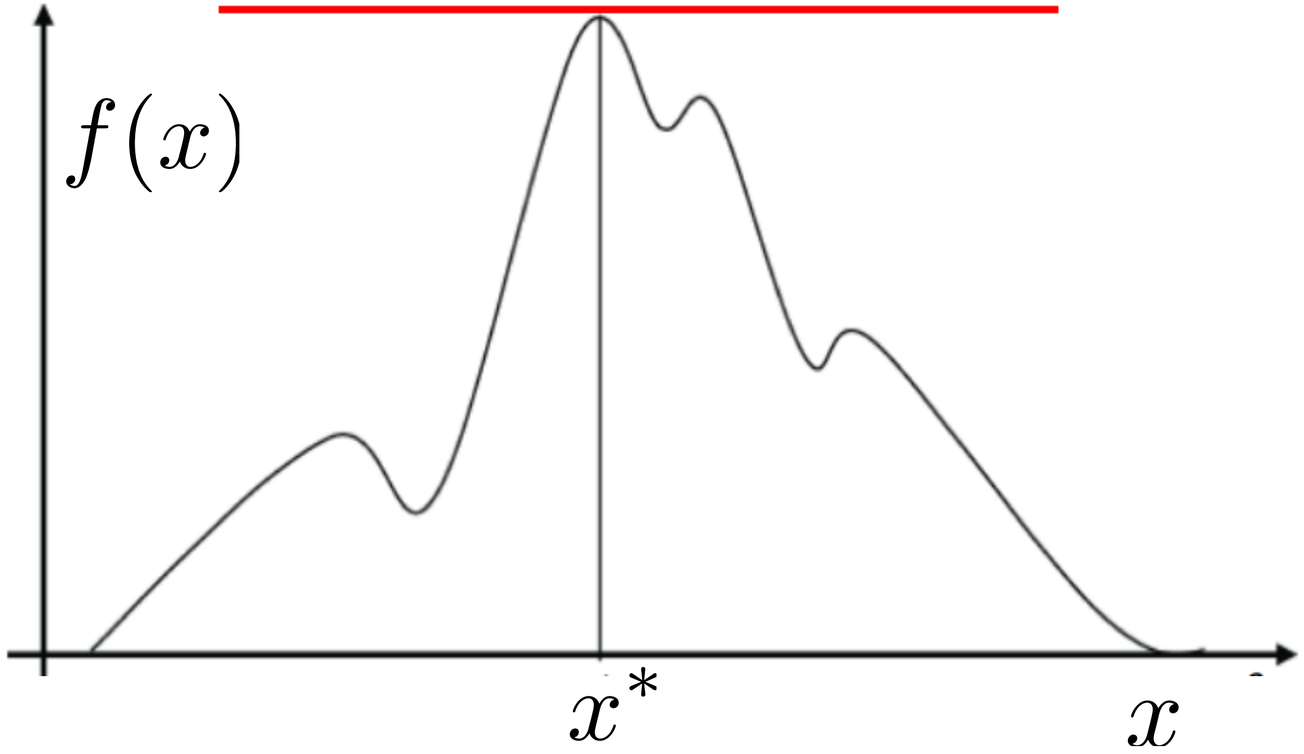


Calculus 101



$$x^* = \text{argmax}_x f(x)$$

Local Extrema Condition



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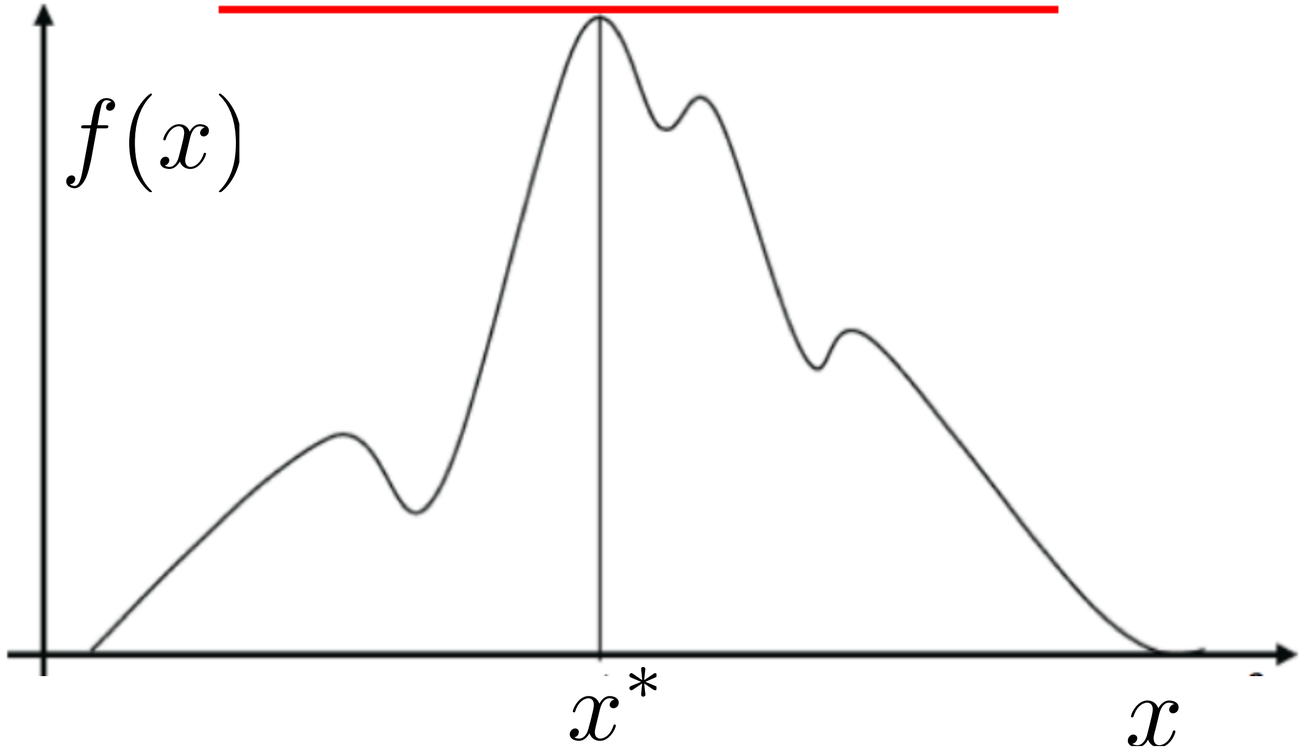

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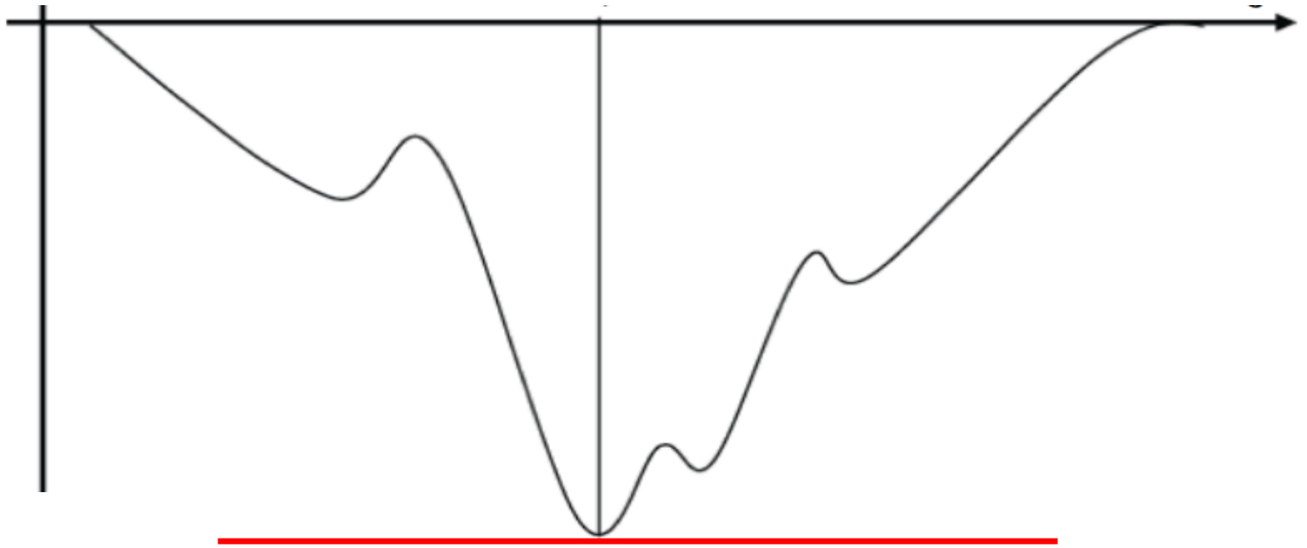

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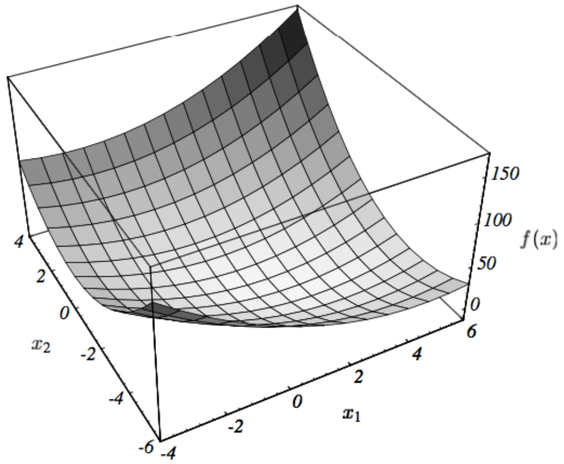
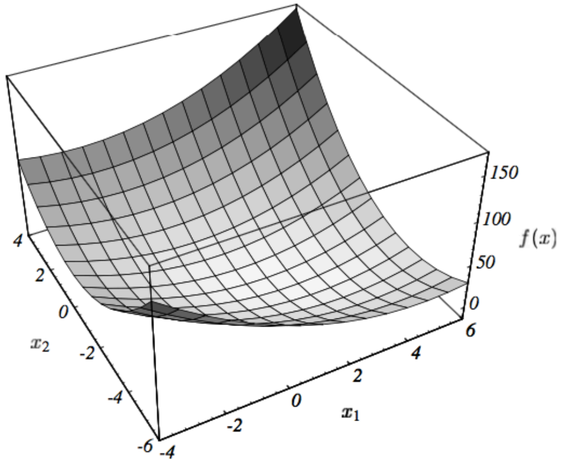
Vector Calculus 101

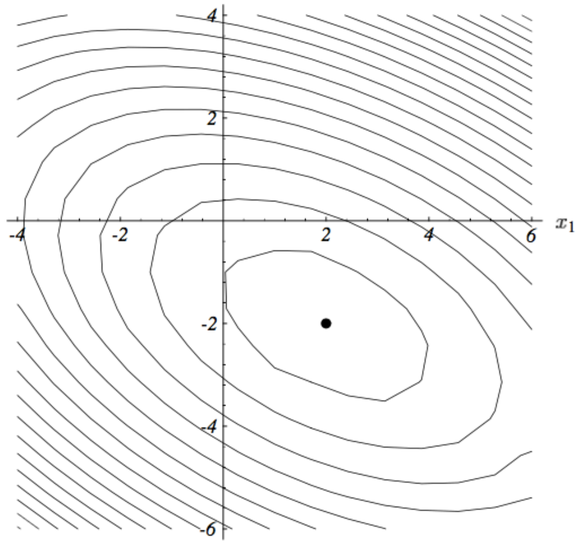


$$f(\mathbf{x})$$

2D function graph

Vector Calculus 101





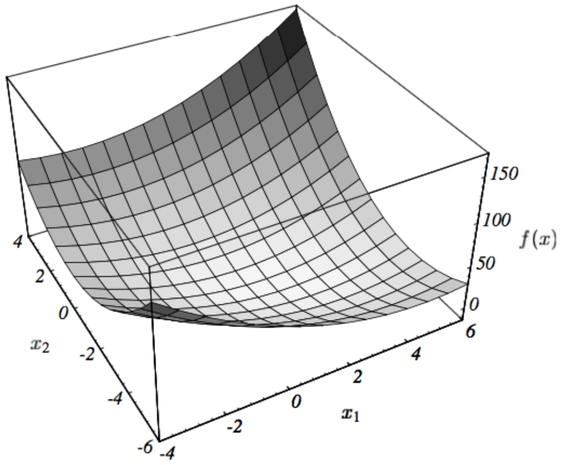
$$f(\mathbf{x})$$

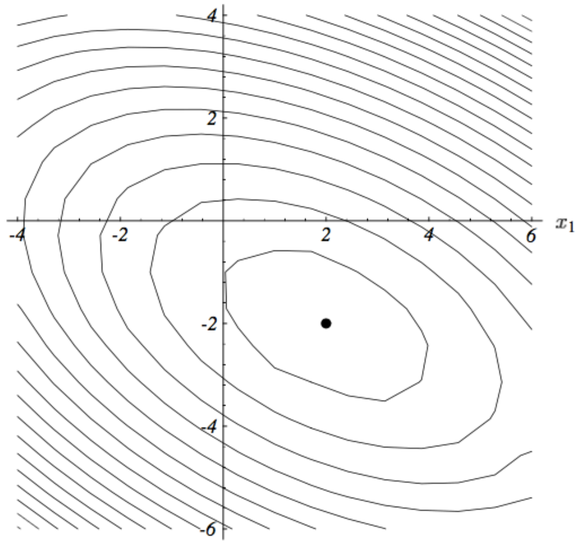
$$f(\mathbf{x}) = c$$

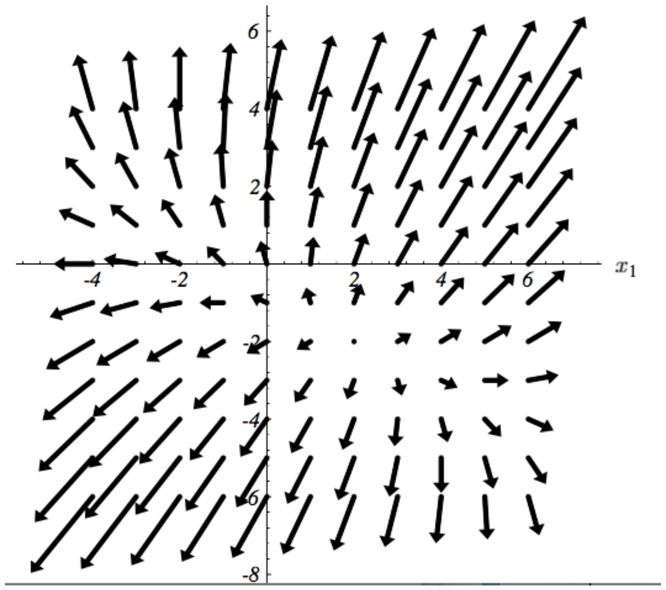
2D function graph

isocontours

Vector Calculus 101







$f(\mathbf{x})$

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$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

2D function graph

isocontours

gradient field

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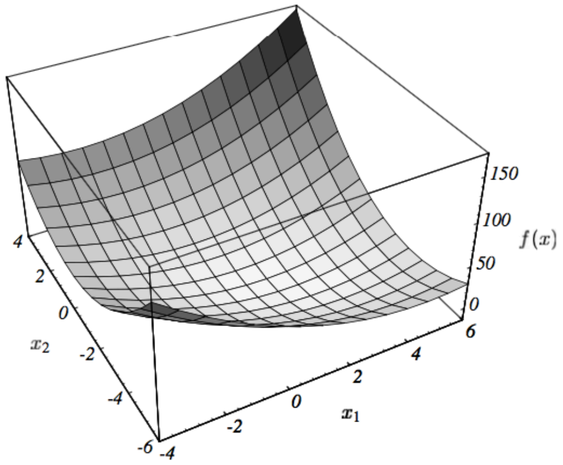

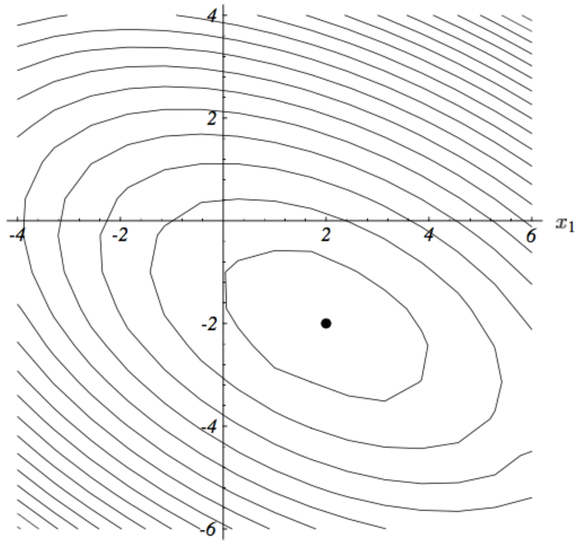

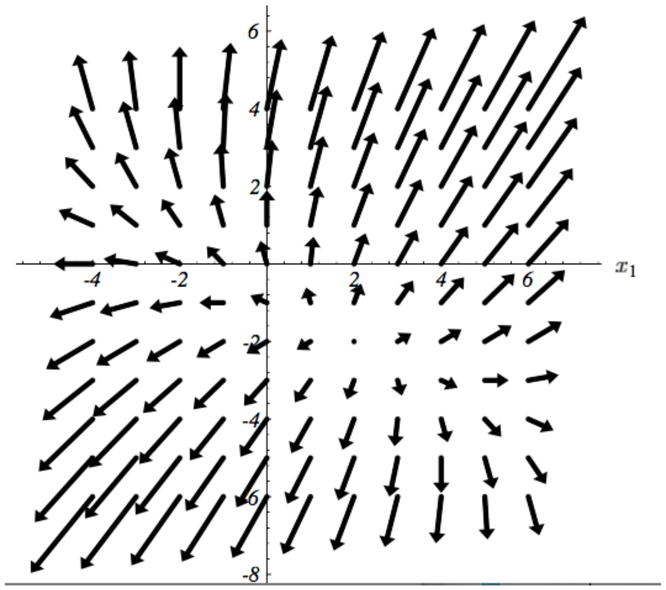

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2D function graph

isocontours

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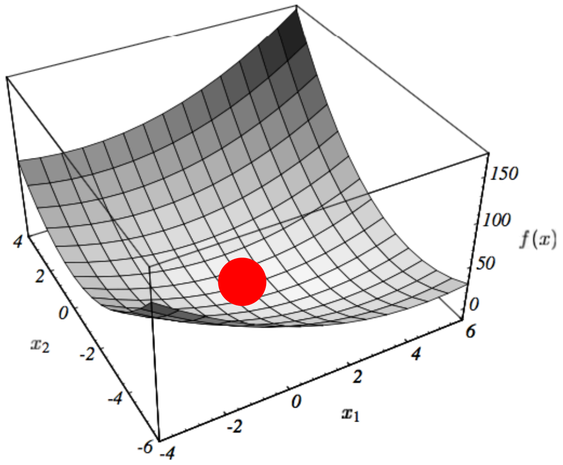
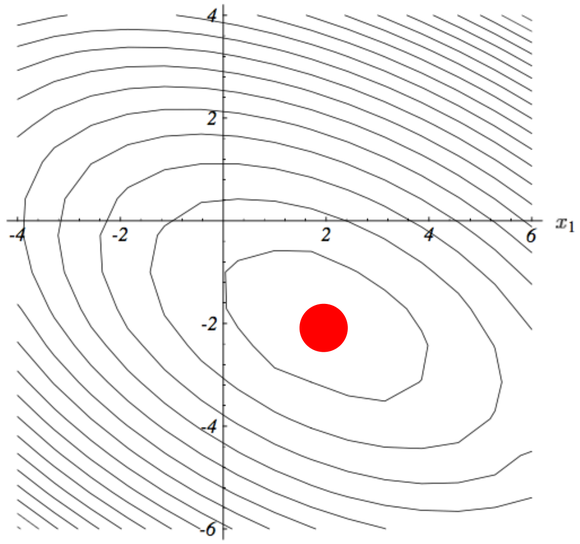
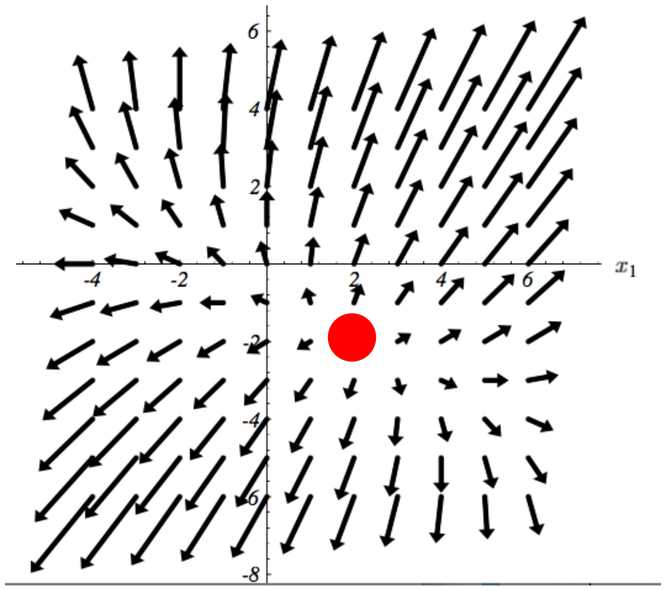
- at minimum of function: $\nabla f(\mathbf{x}) = \mathbf{0}$

Vector Calculus 101







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Line Fitting

$$L(w_0, w_1) = \sum_{i=1}^{N} \left[y^i - \left(w_0 x_0^i + w_1 x_1^i \right) \right]^2$$

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Line Fitting

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$$= -2 \sum_{i=1}^{N} (y^i - (w_0 x_0^i + w_1 x_1^i)) x_0^i$$

$$z^i = y^i - (w_0 x_0^i + w_1 x_1^i)$$

Line Fitting (continued)

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Line Fitting (continued)

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$$= -2 \sum_{i=1}^{N} \left(y^i x_0^i - w_0 x_0^i x_0^i - w_1 x_1^i x_0^i \right)$$

$$\frac{\partial L(w_0, w_1)}{\partial w_0} = 0$$

Line Fitting (continued)

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2x2 system of equations

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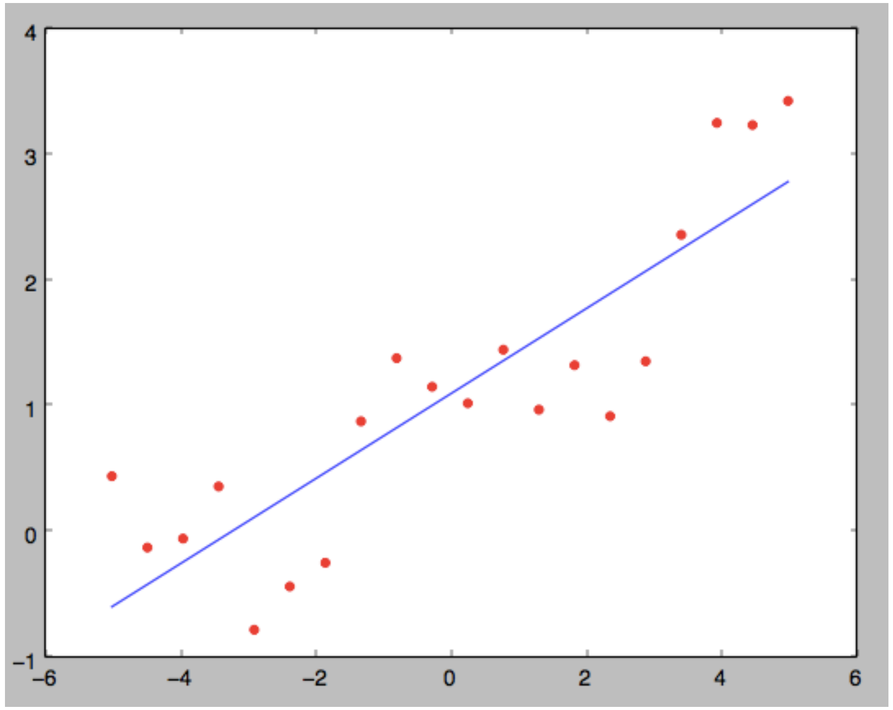
Normal Equation

$$\mathbf{w} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Code Example



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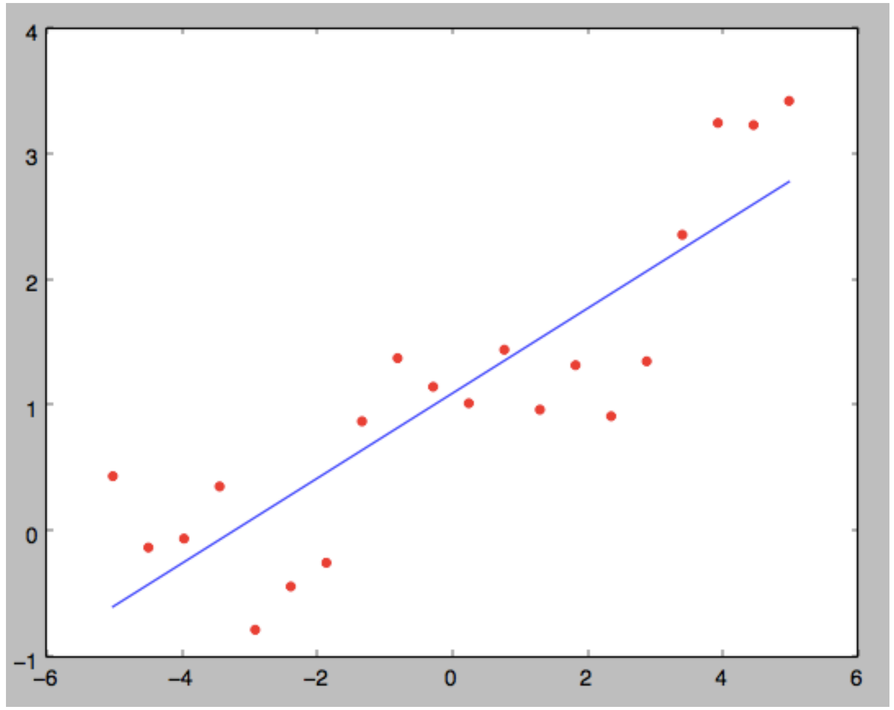


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x  = np.linspace(-5,5,20)
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# create the design matrix: the x data, and add a column of ones for the constant term
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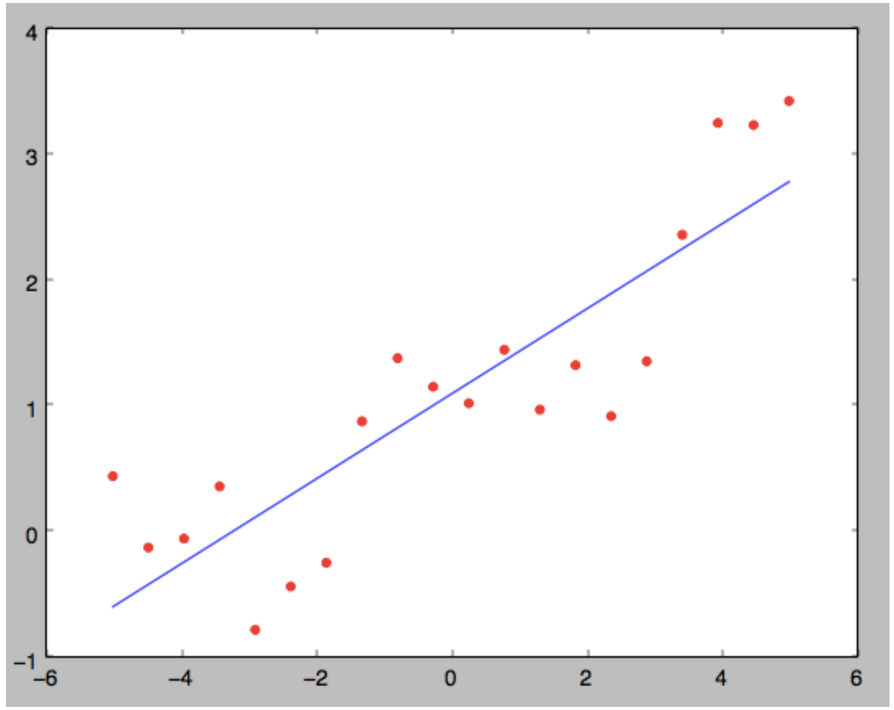


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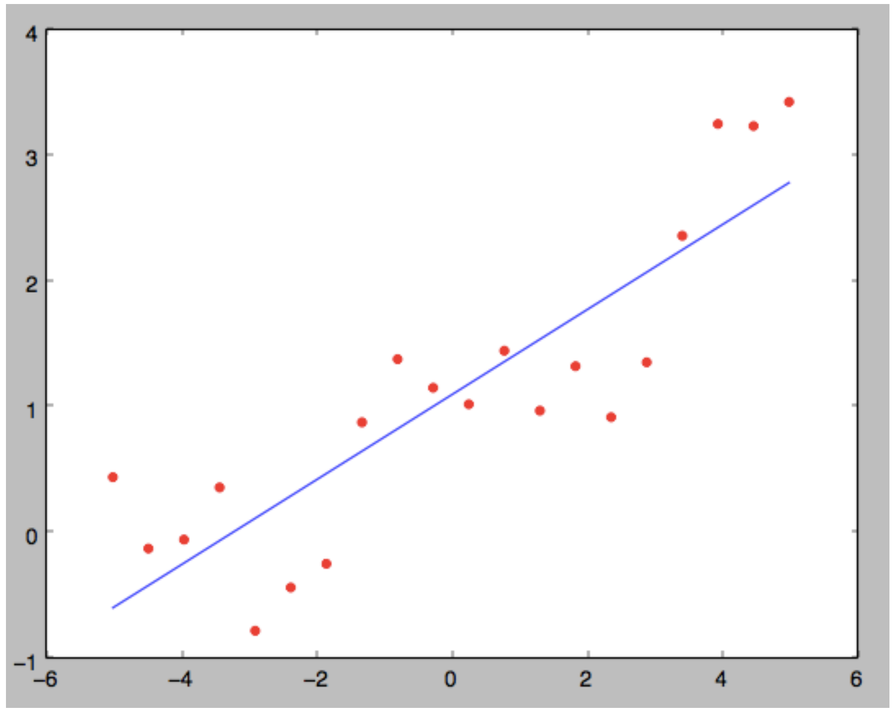


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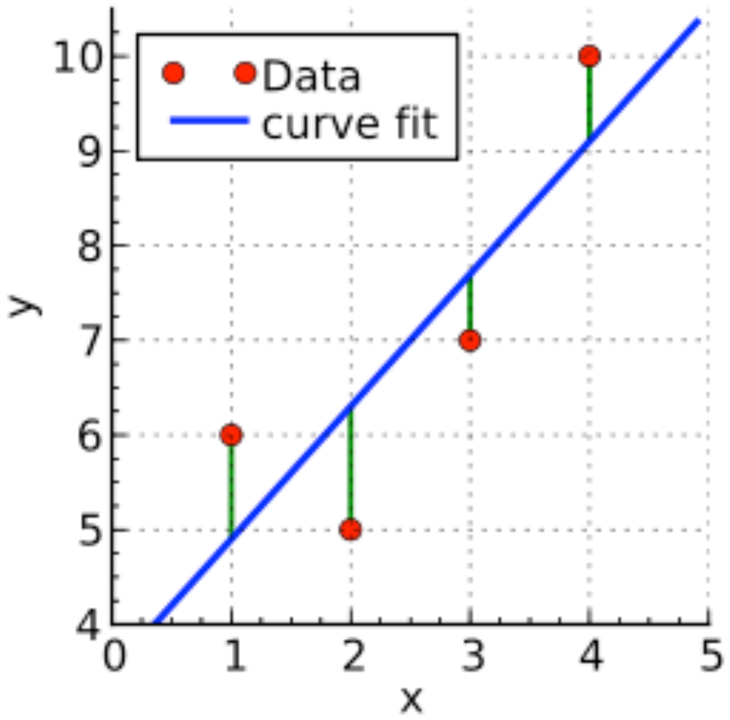
Linear Regression (Line/Plane Fitting)



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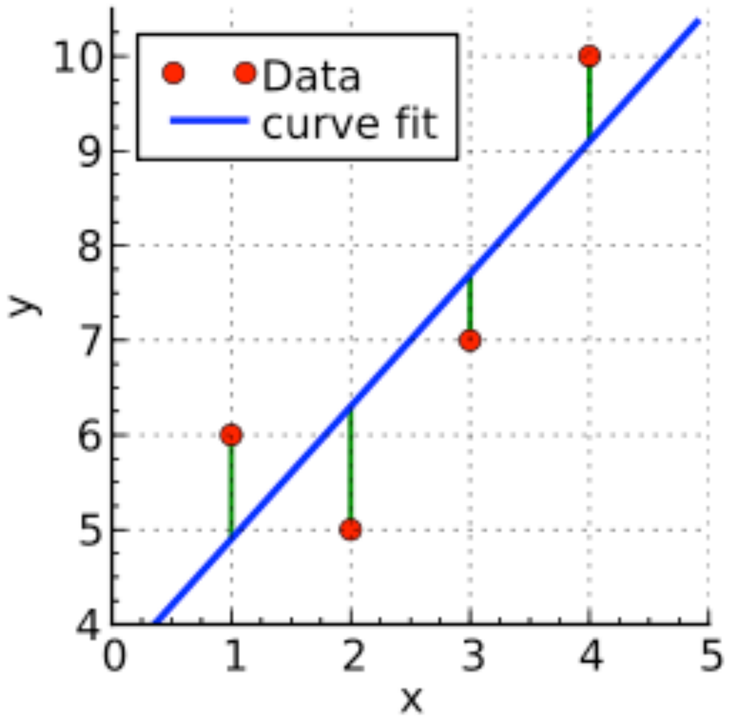

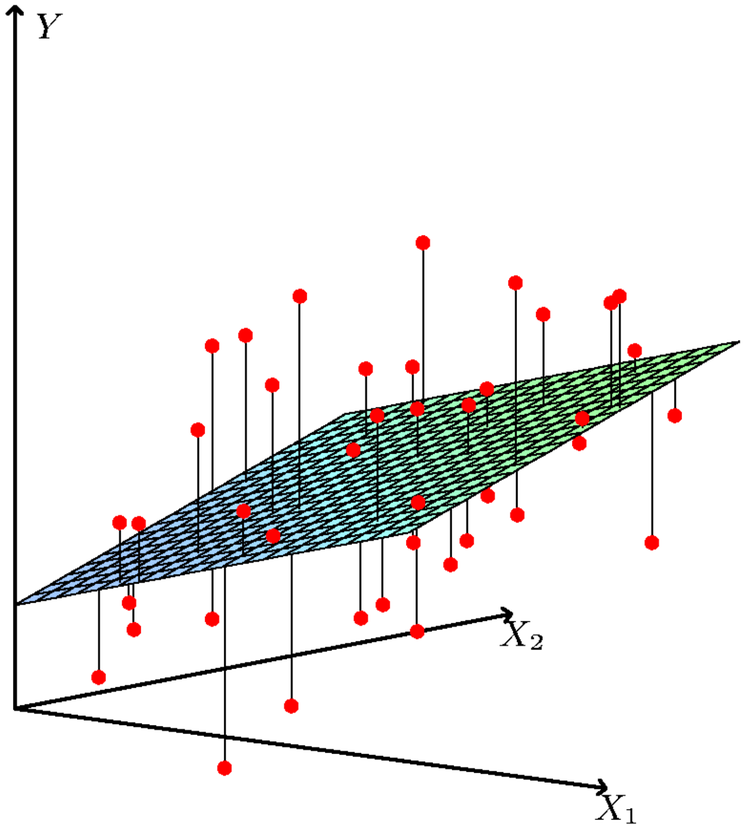

LS Solution for Regression

$$L(\mathbf{w}) = \sum_{i=1}^{N} (y^i - \mathbf{w}^T \mathbf{x}^i)^2 = \sum_{i=1}^{N} (\epsilon^i)^2$$

$$L(\mathbf{w}) = \begin{bmatrix} \epsilon^1 & \epsilon^2 & \dots & \epsilon^N \end{bmatrix} \begin{bmatrix} \epsilon^1 \\ \epsilon^2 \\ \vdots \\ \epsilon^N \end{bmatrix}$$

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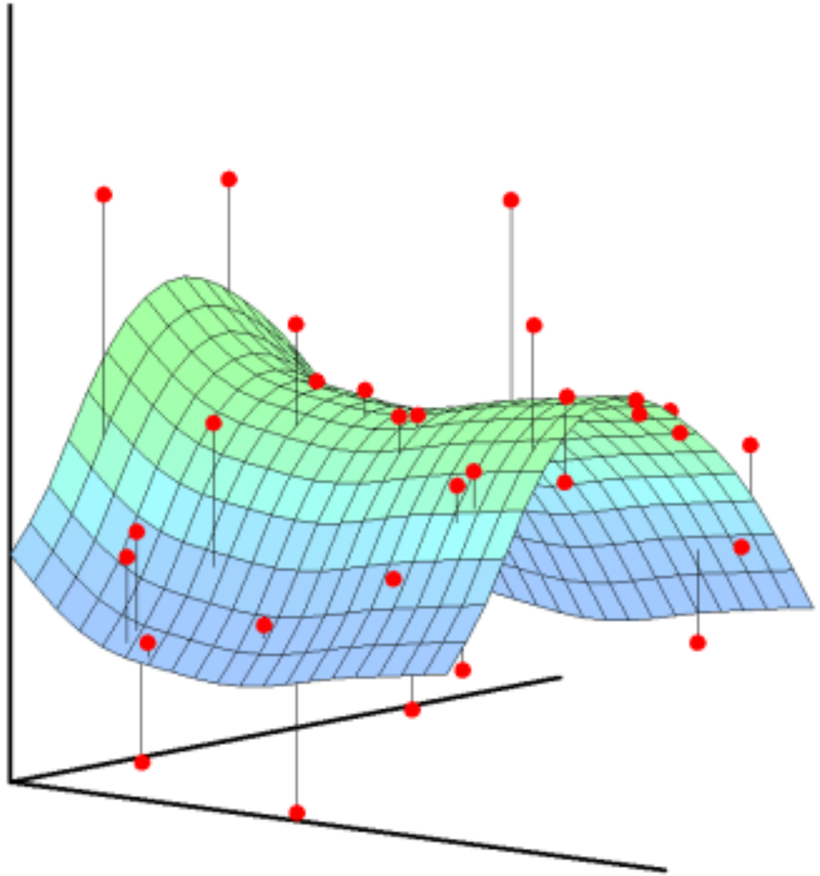
Generalized Linear Regression



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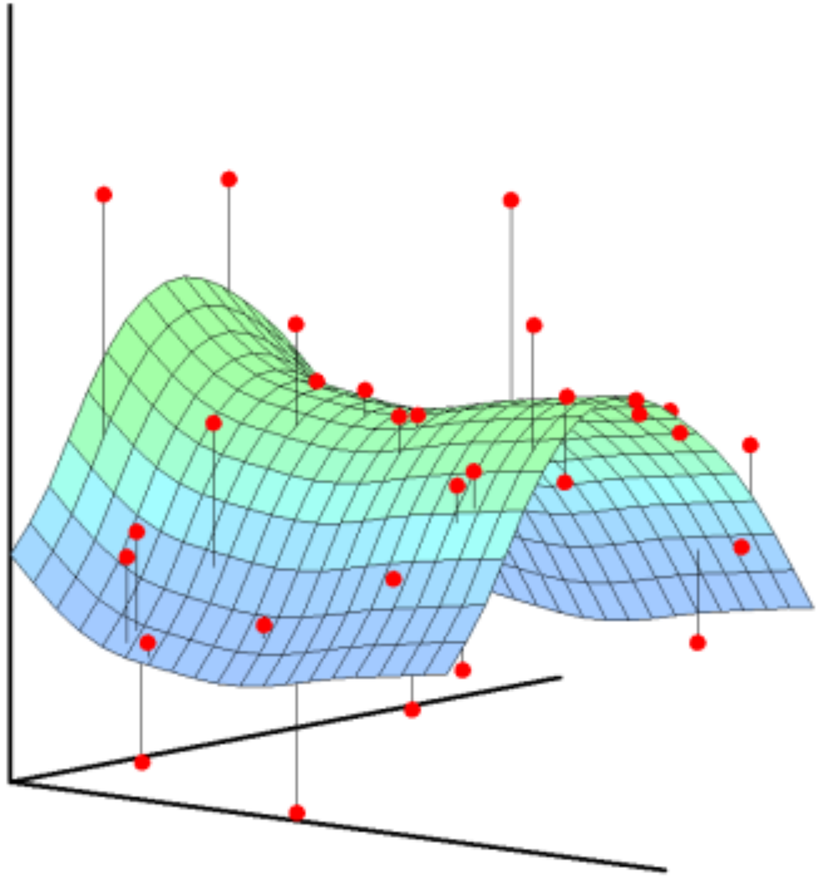

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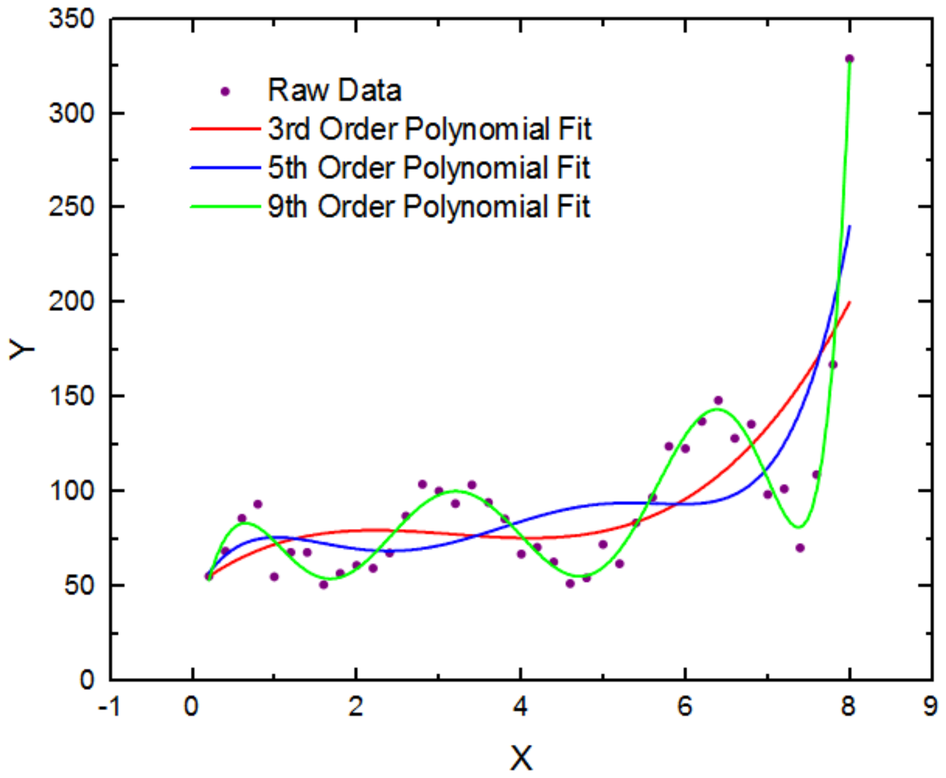
known nonlinearity

1D Example: k-th Degree Polynomial Fitting

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{bmatrix} 1 \\ x \\ \vdots \\ (x)^K \end{bmatrix}$$



$$\langle \mathbf{w}, \boldsymbol{\phi}(x) \rangle = w_0 + w_1 x + \ldots + w_k (x)^K$$

Generalized Linear Regression

$$L(\mathbf{w}) = \sum_{i=1}^{N} (y^i - \mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}^i))^T = \sum_{i=1}^{N} (\epsilon^i)^2$$

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$$\underset{\text{Nx1}}{\begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^N \end{bmatrix}} = \underset{\text{NxM}}{\begin{bmatrix} \boldsymbol{\phi}(\mathbf{x}^1)^T \\ \hline \boldsymbol{\phi}(\mathbf{x}^2)^T \\ \hline \vdots \\ \hline \boldsymbol{\phi}(\mathbf{x}^N)^T \end{bmatrix}} \underset{\text{Mx1}}{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_M \end{bmatrix}} + \underset{\text{Nx1}}{\begin{bmatrix} \epsilon^1 \\ \epsilon^2 \\ \vdots \\ \epsilon^N \end{bmatrix}}$$

$$\boldsymbol{\phi}(\mathbf{x}) : \mathbb{R}^D \rightarrow \mathbb{R}^M$$

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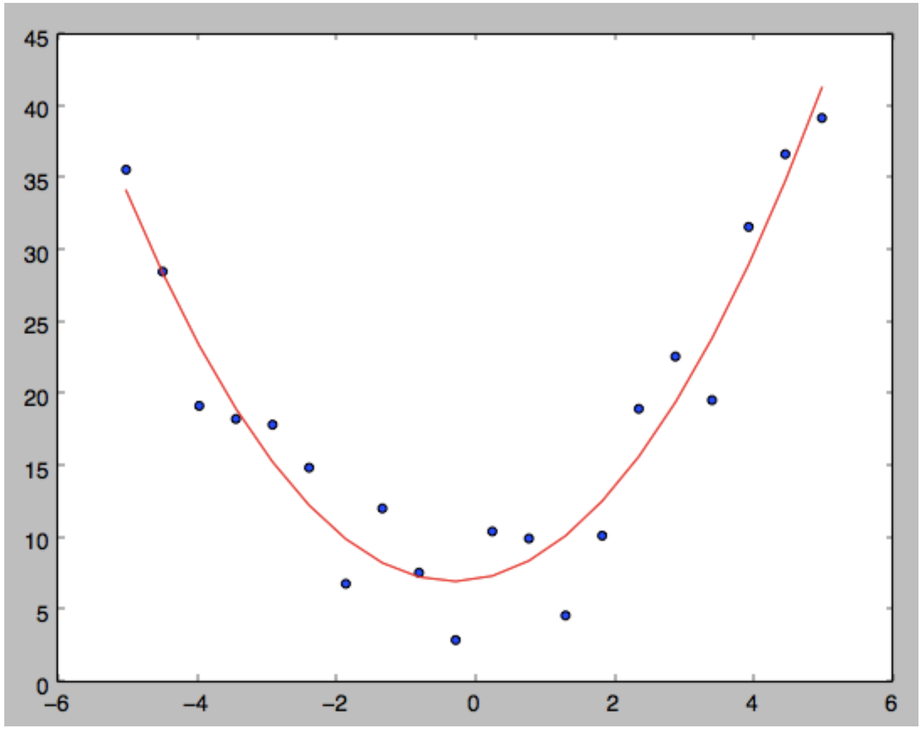


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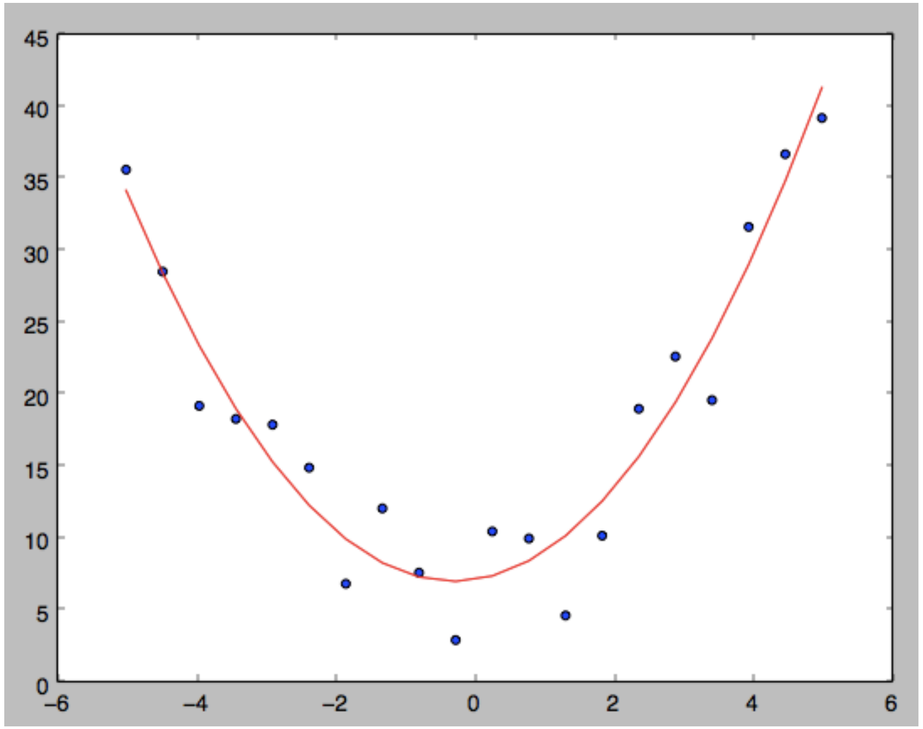


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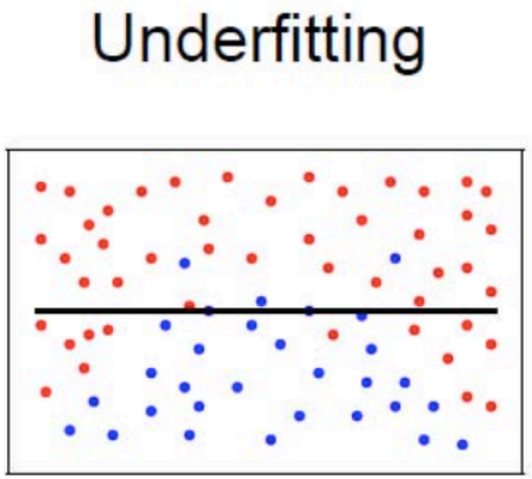
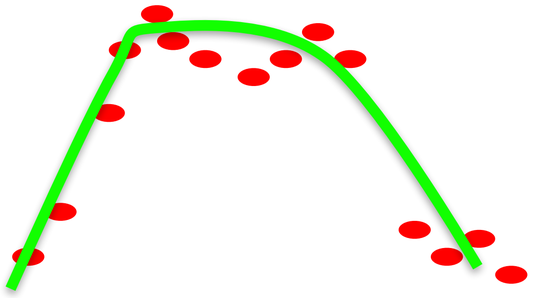
Underfitting vs. Overfitting

classification



regression



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Underfitting

Overfitting

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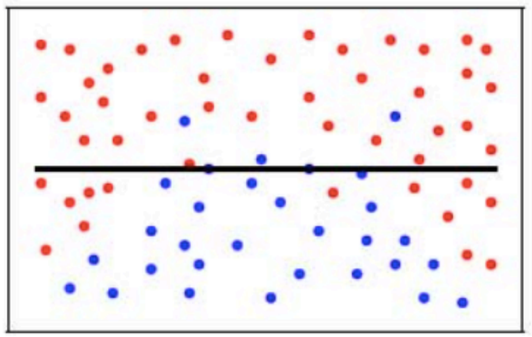

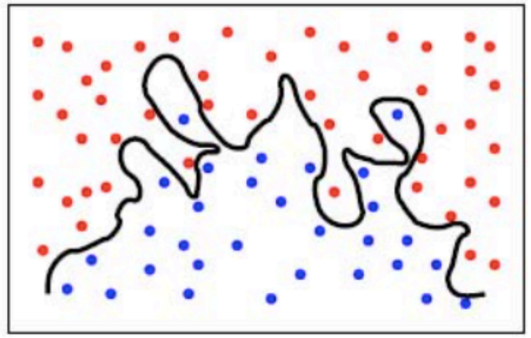

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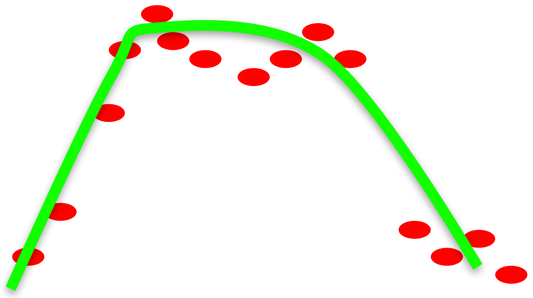

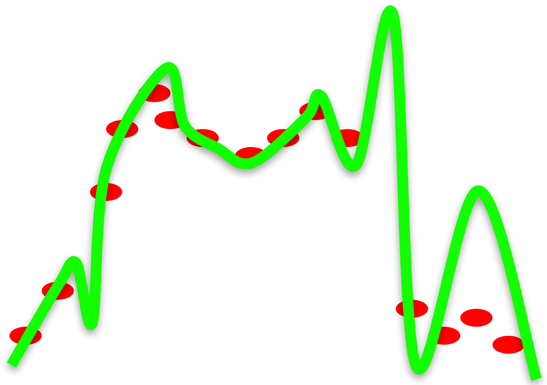

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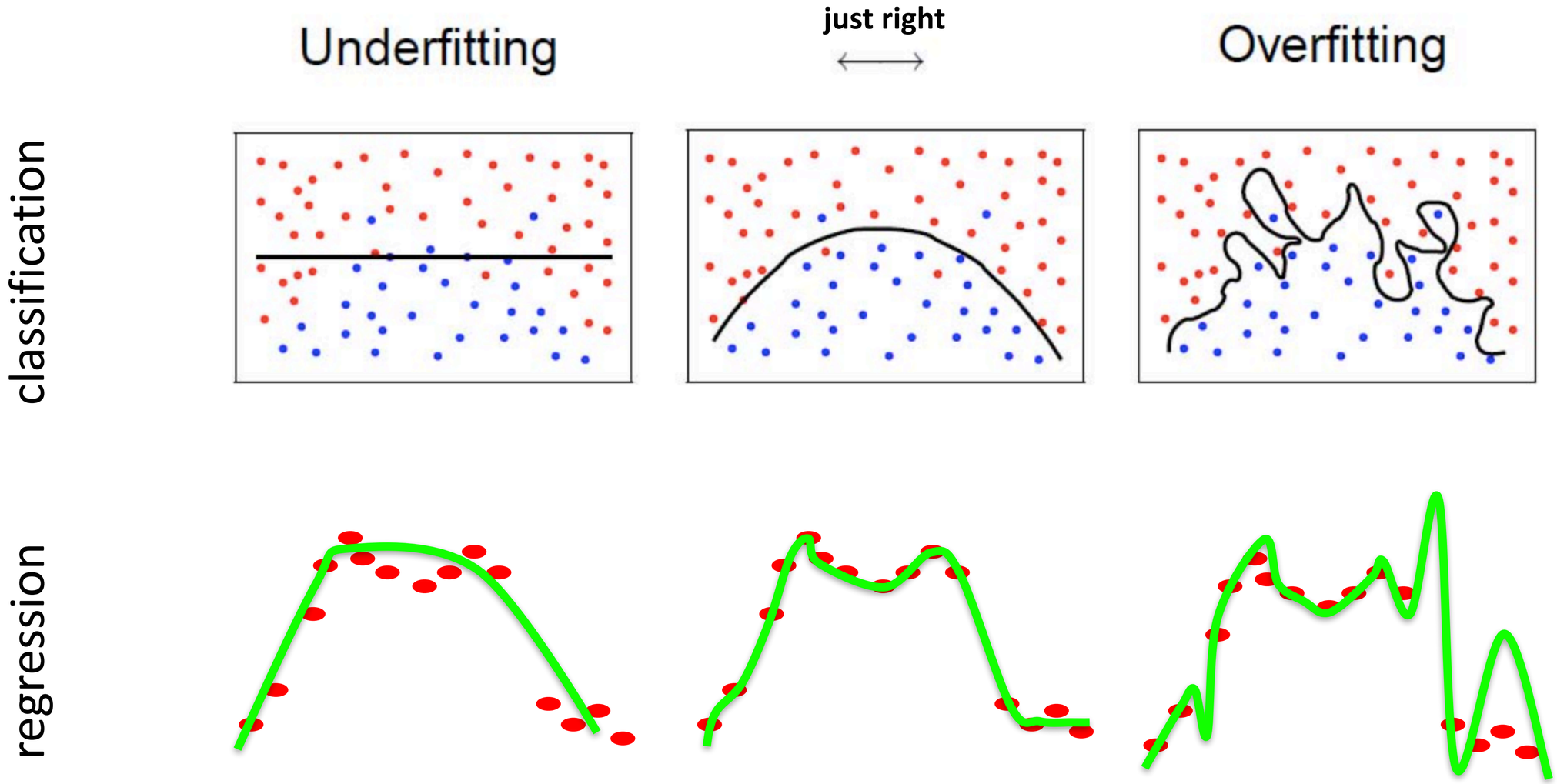

Tuning Model's Complexity

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but can "**overtrain**" and have poor performance on the test set ("variance").

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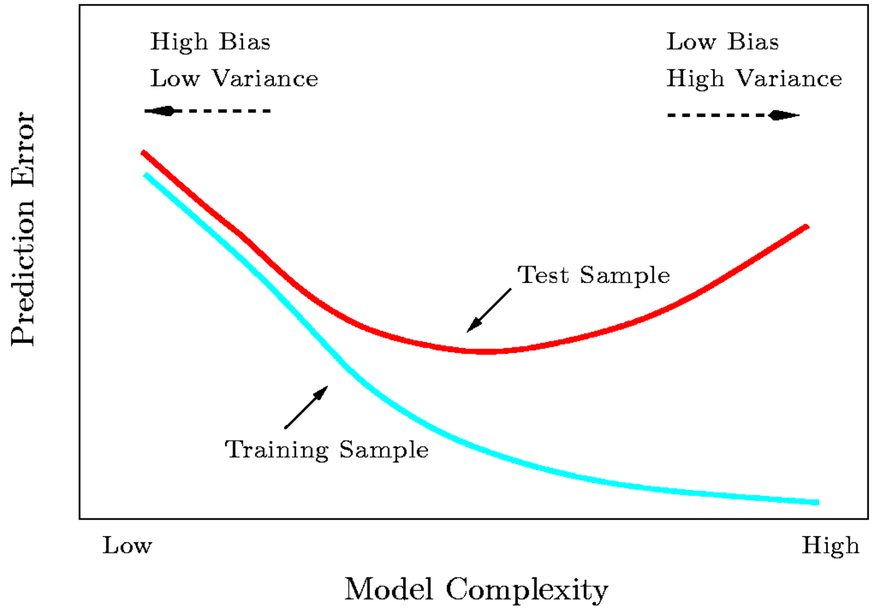

Regularized Linear Regression

$$\boldsymbol{\epsilon} = \mathbf{y} - \boldsymbol{\Phi}\mathbf{w}$$

residual vector

$$L(\mathbf{w}) = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon}$$

linear regression: minimize model error

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$$\boldsymbol{\epsilon} = \mathbf{y} - \boldsymbol{\Phi}\mathbf{w}$$ residual vector

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Complexity term: (regularizer) $$R(\mathbf{w}) \doteq \|\mathbf{w}\|_2^2 = \mathbf{w}^T \mathbf{w}$$

Regularized Linear Regression

$$\boldsymbol{\epsilon} = \mathbf{y} - \boldsymbol{\Phi}\mathbf{w}$$ residual vector

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"data fidelity"

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Least Squares Solution

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\begin{aligned}
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as before, for linear regression

identity matrix

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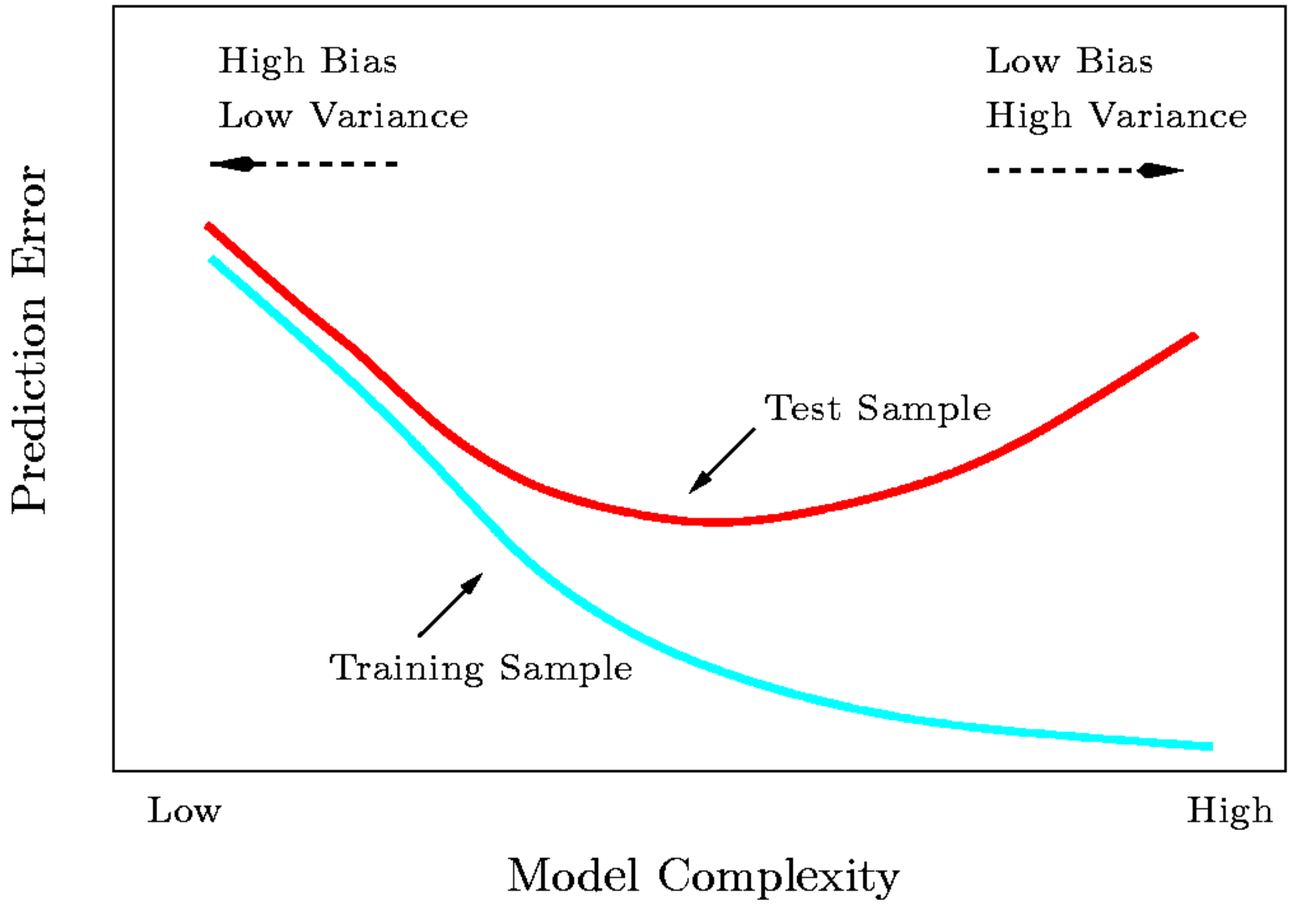
λ: "hyperparameter"

Bias-Variance Tradeoff (function of λ)



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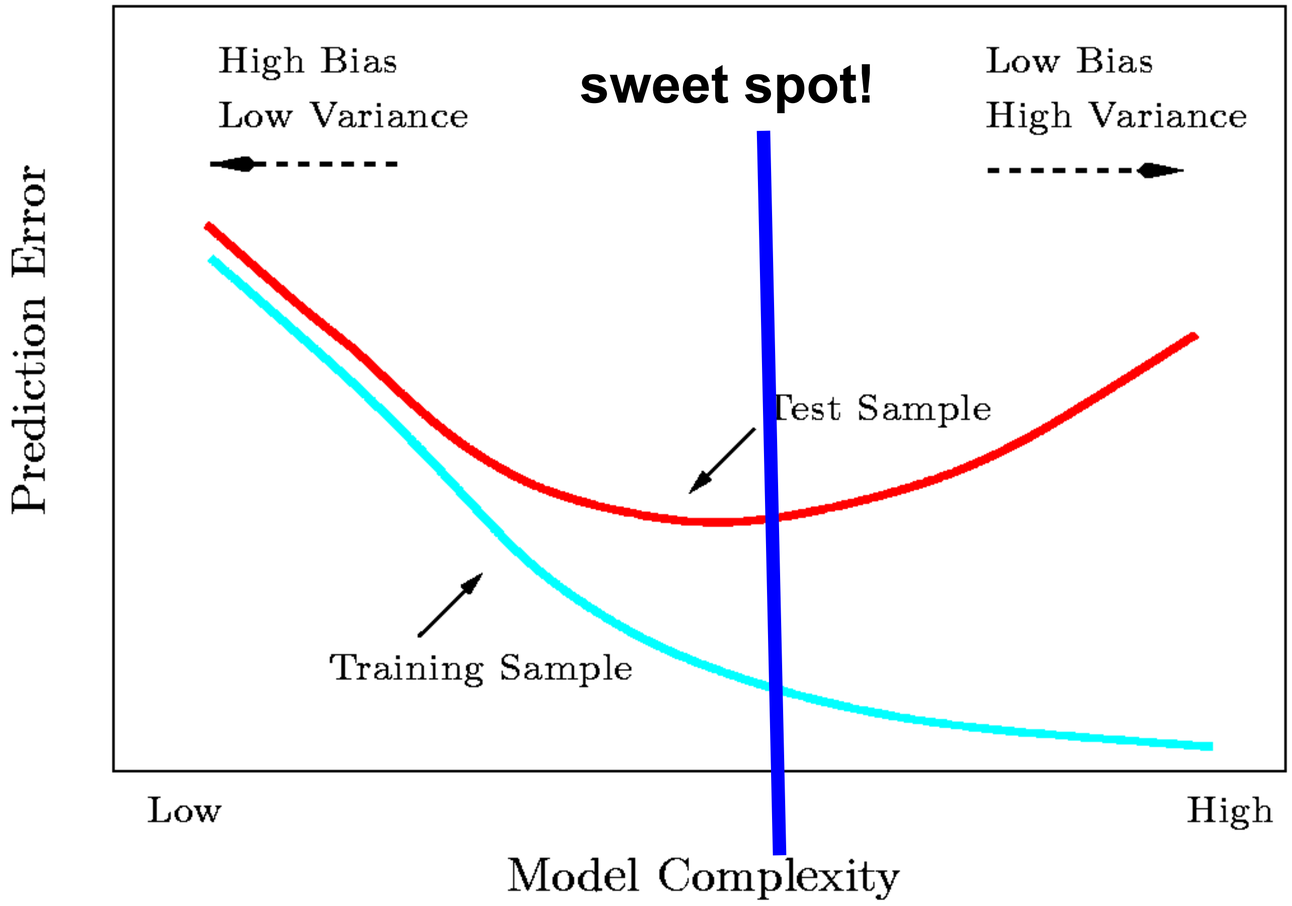
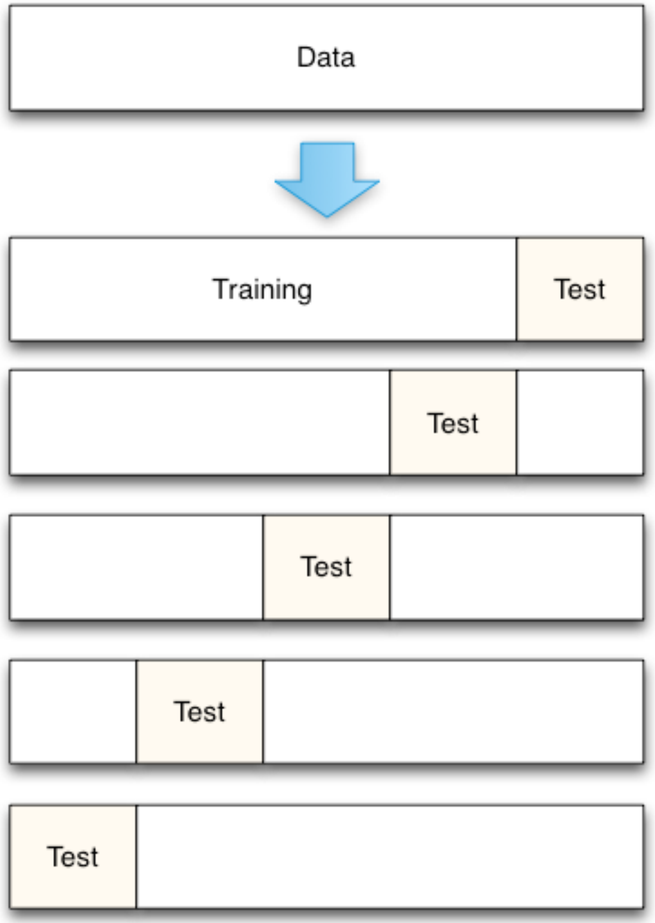


Selecting λ with Cross-validation



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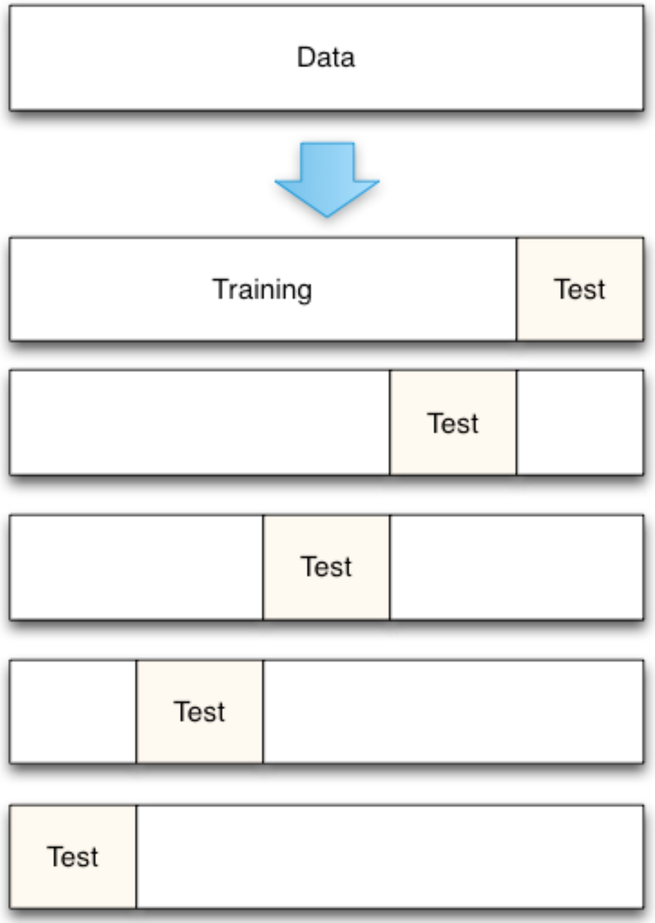

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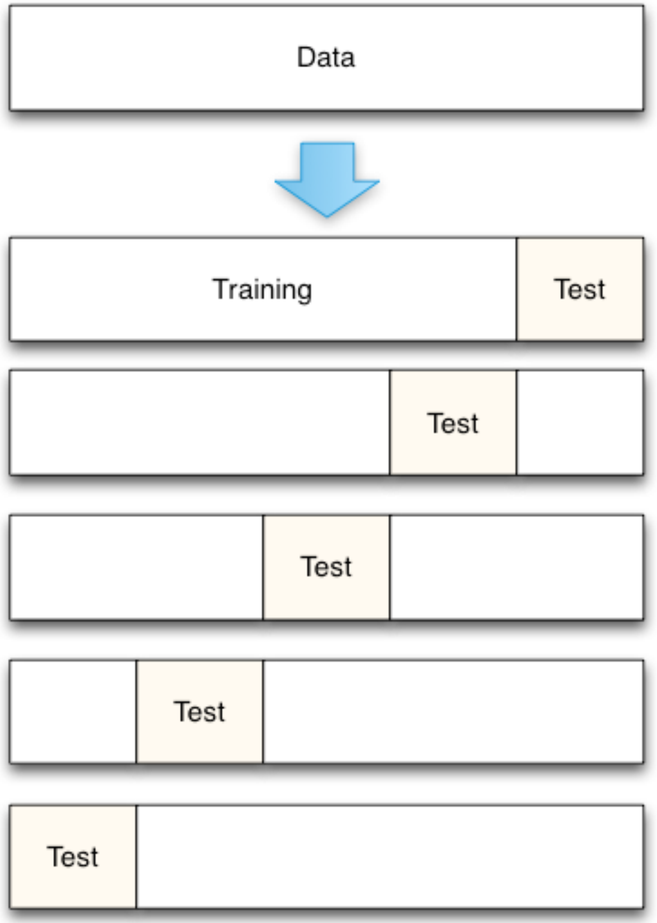

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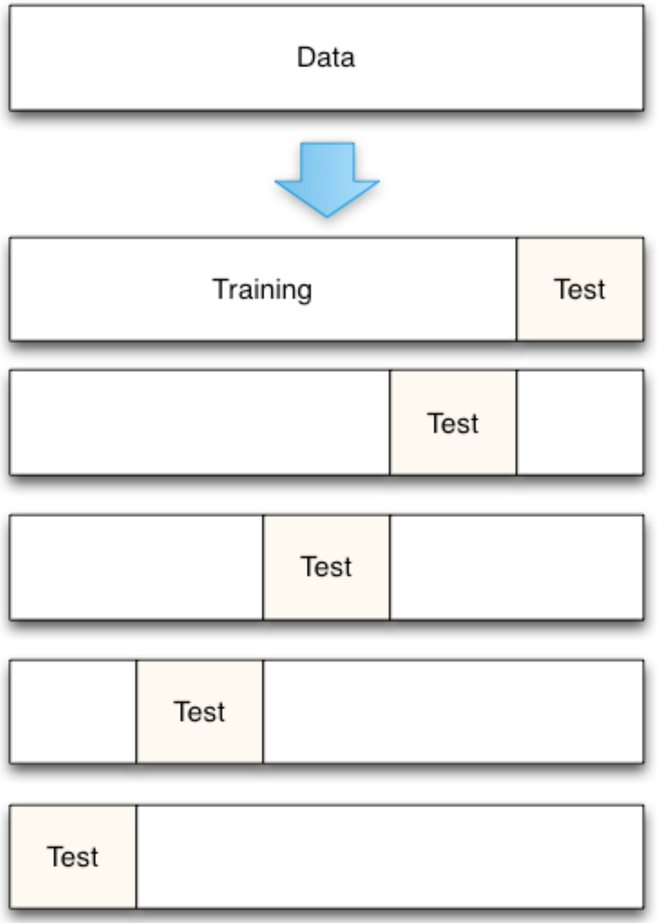

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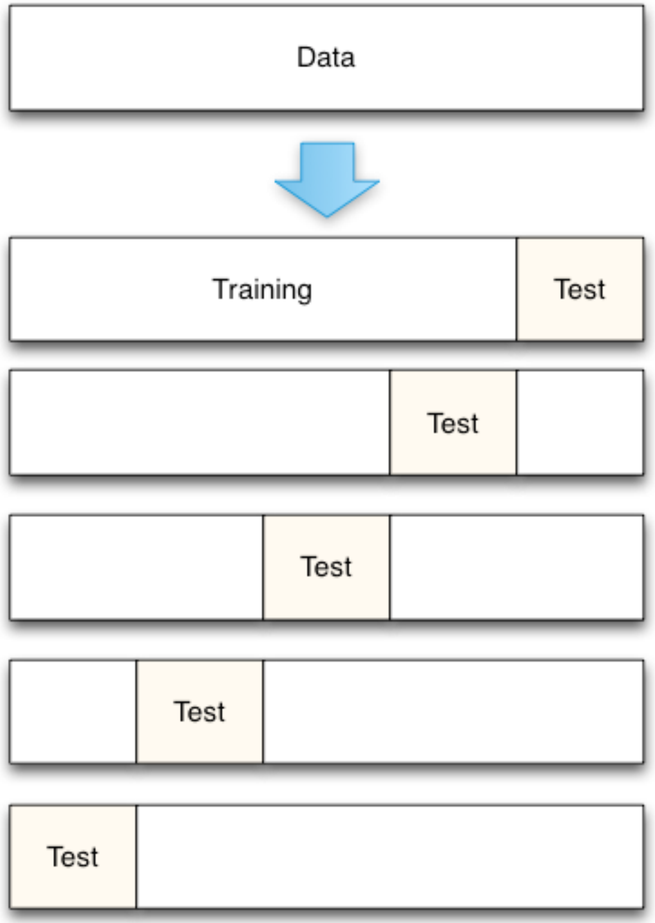

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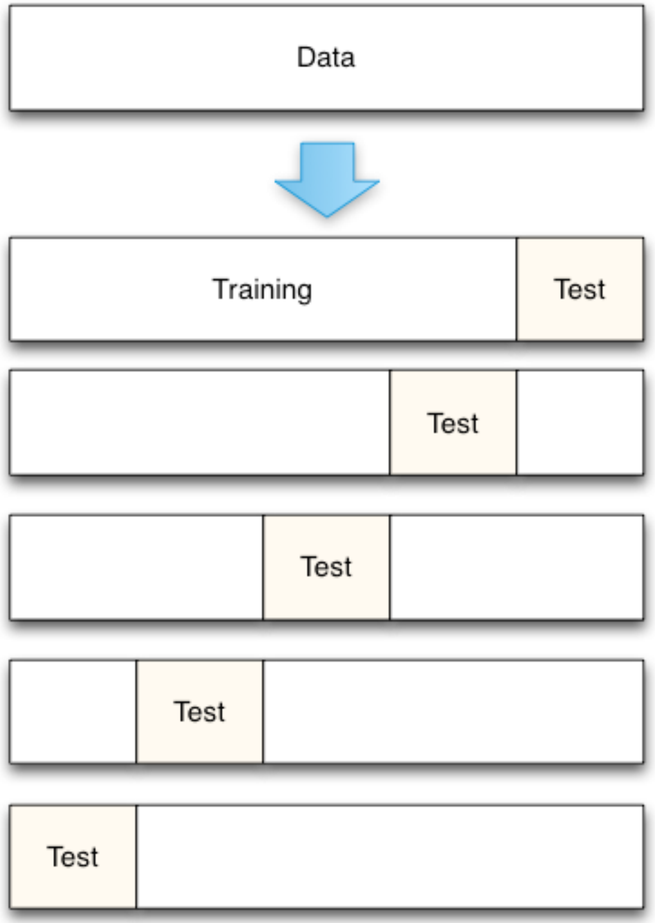
Least glorious, most effective of all methods



Form of posterior distribution

Bernoulli-type conditional distribution

$$\left.\begin{aligned} P(Y=1|X=\mathbf{x};\mathbf{w}) &= f(\mathbf{x},\mathbf{w}) \\ P(Y=0|X=\mathbf{x};\mathbf{w}) &= 1-f(\mathbf{x},\mathbf{w}) \end{aligned}\right\} \rightarrow$$

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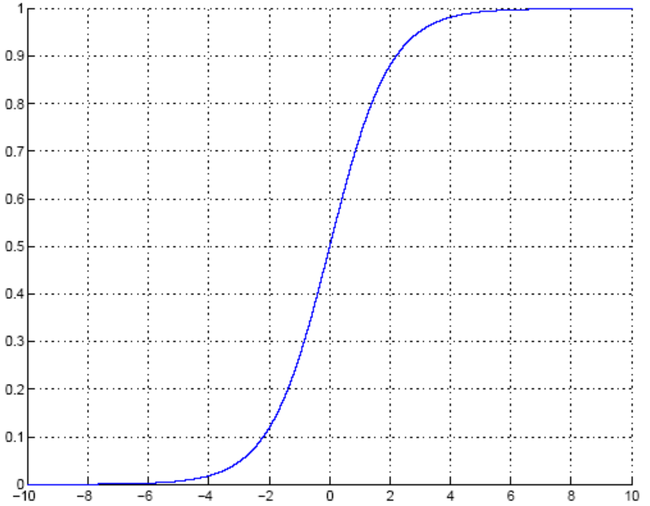

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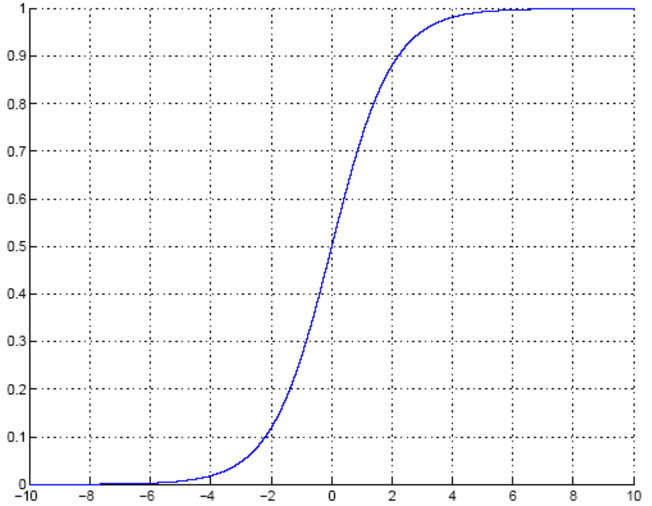

"squashing function":

$$-\infty \rightarrow 0$$
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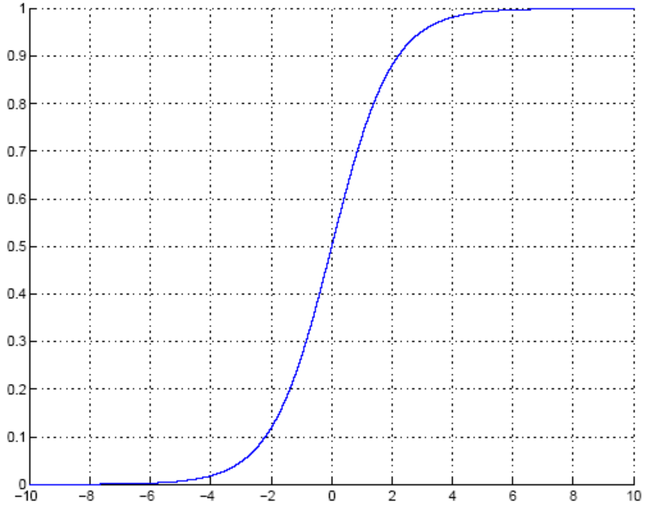

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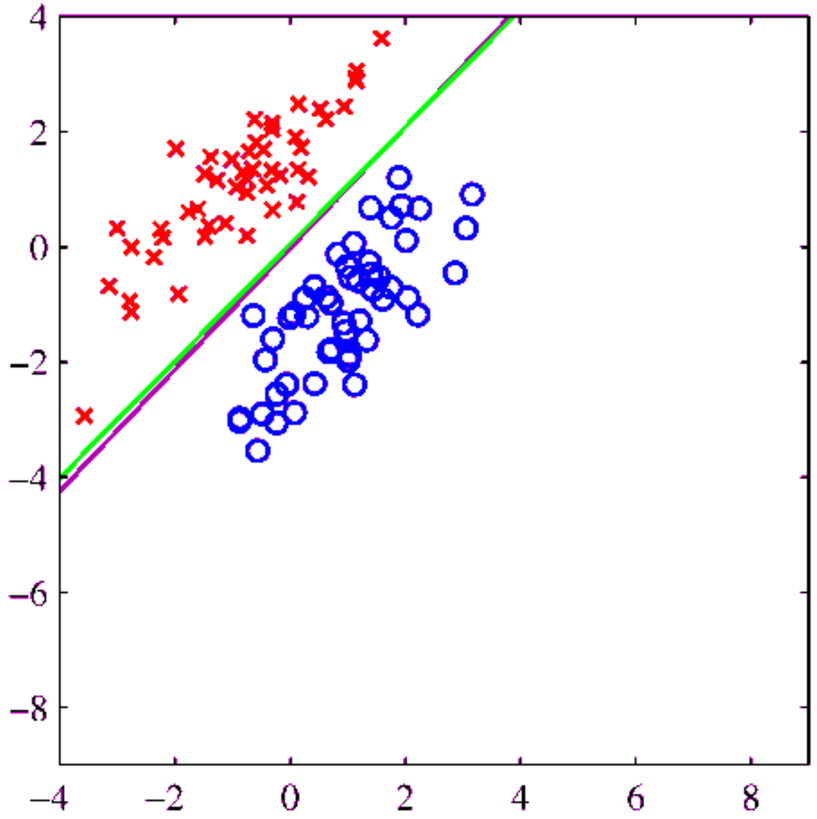

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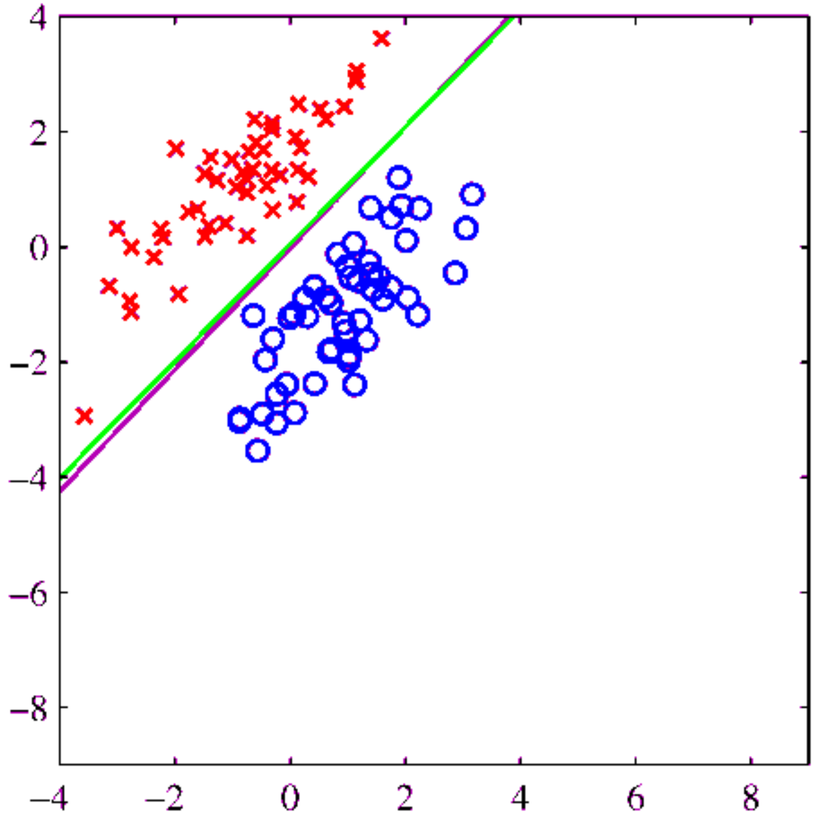

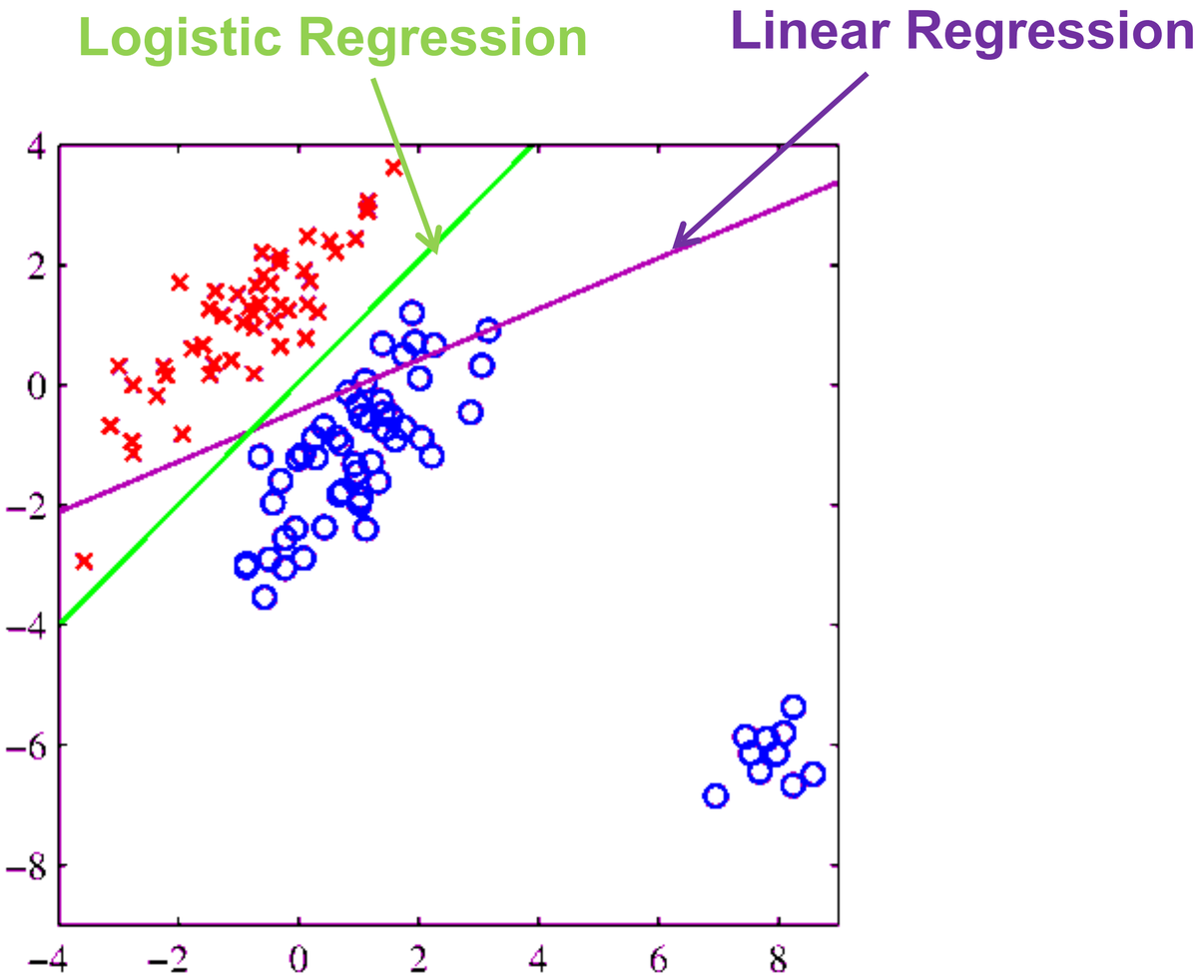

From Two to Many

- How about multi-class classification?

Multiple Classes & Linear Regression

C classes: one-of-c coding (or one-hot encoding)

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4 classes, i-th sample is in 3rd class:

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where $\mathbf{y}_c = \begin{bmatrix} y_c^1 \\ \vdots \\ y_c^N \end{bmatrix}$

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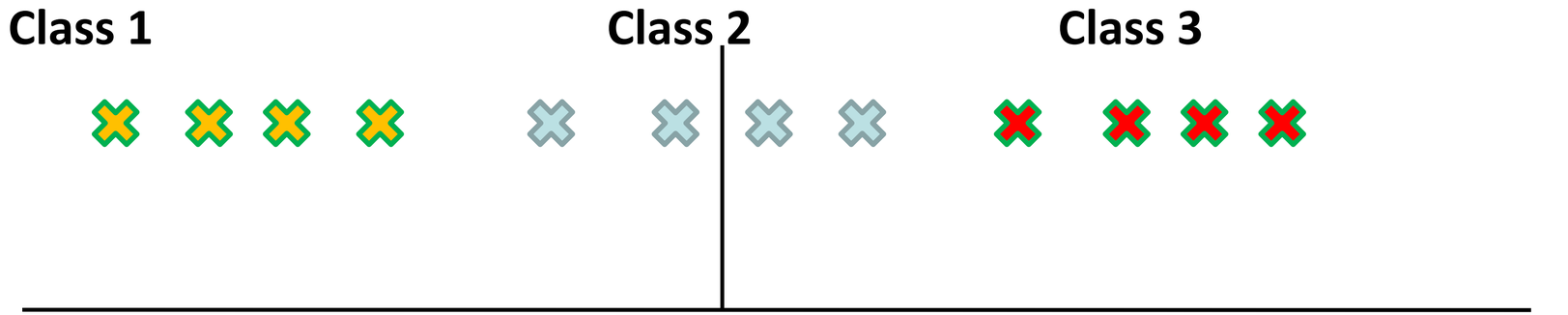
Least squares fit (decouples per class):

$$\mathbf{w}_c^* = (\mathbf{X}^T\mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}_c$$

Linear Regression Masking Problem



One linear discriminant per class: $s_c(\mathbf{x}) = \mathbf{w}_c^T \mathbf{x}$

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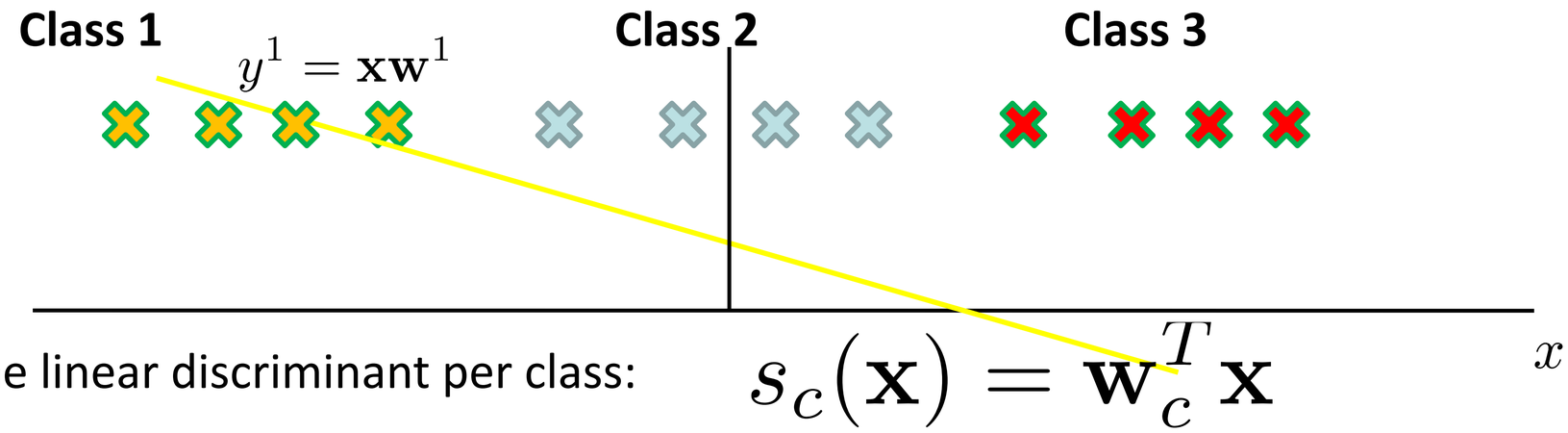

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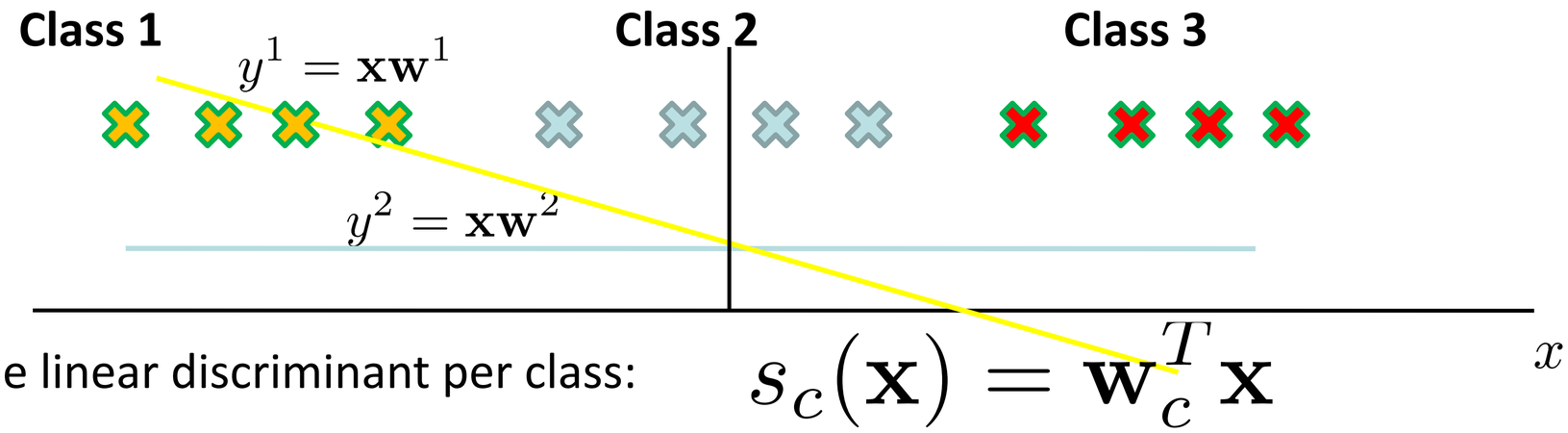

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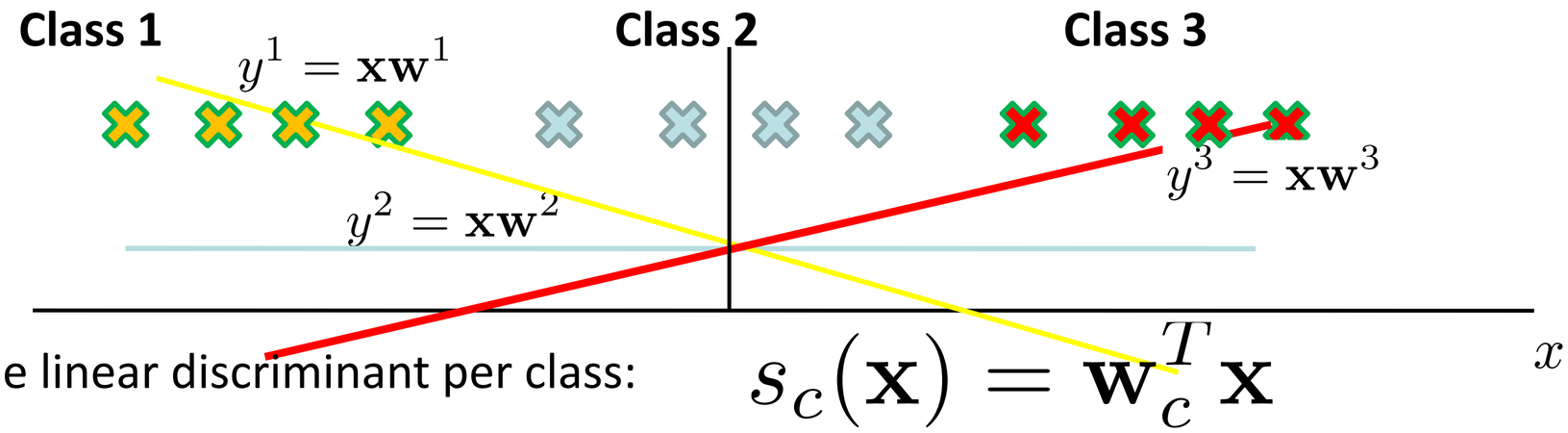

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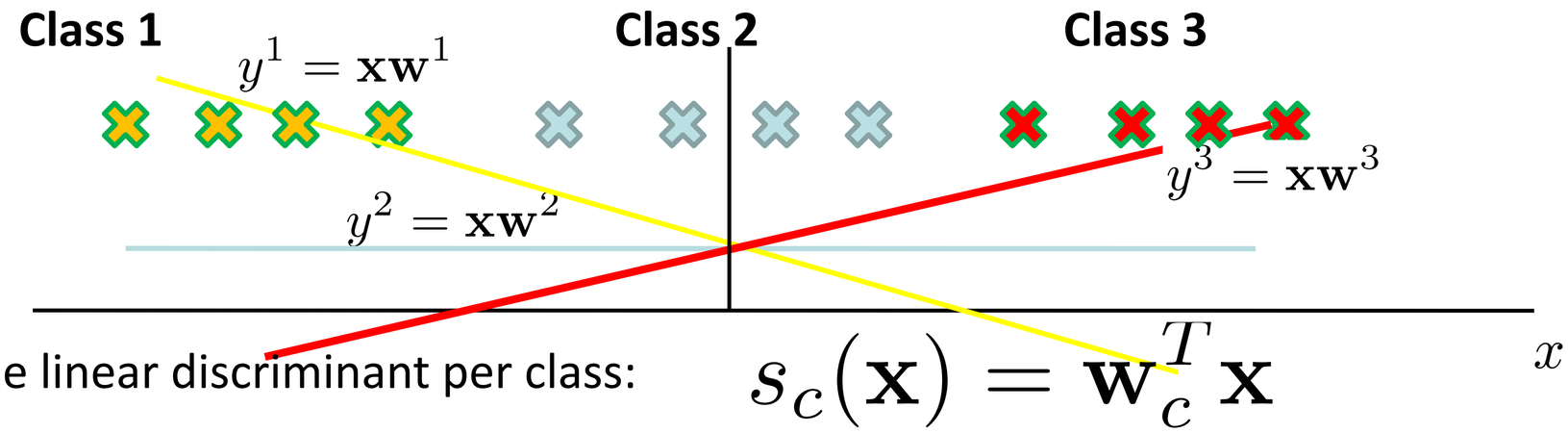

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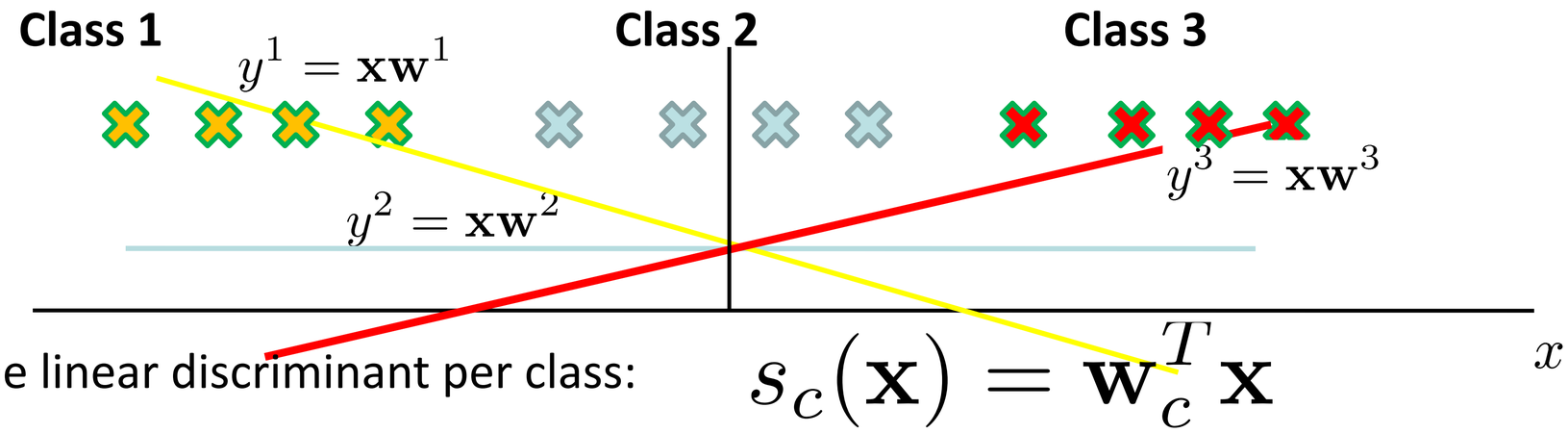

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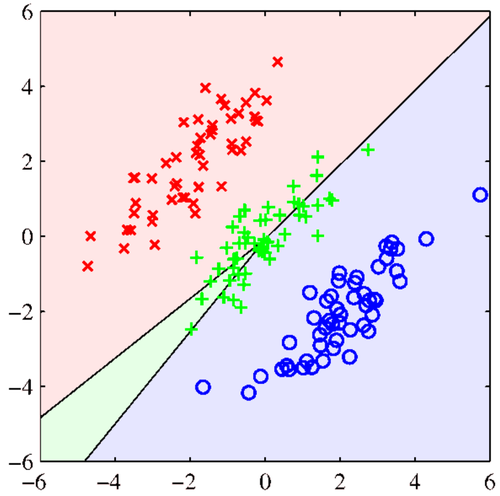
2D version:



Multiple classes & Logistic regression

Soft maximum (softmax) of competing classes:

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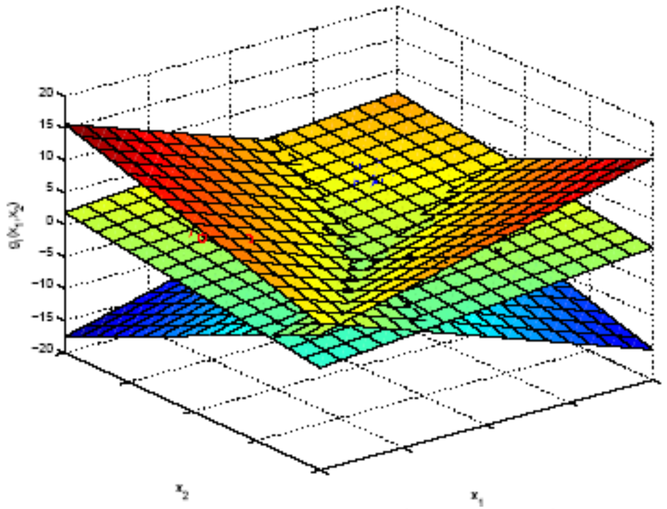

Discriminants (inputs)

Multiple classes & Logistic regression

Soft maximum (softmax) of competing classes:

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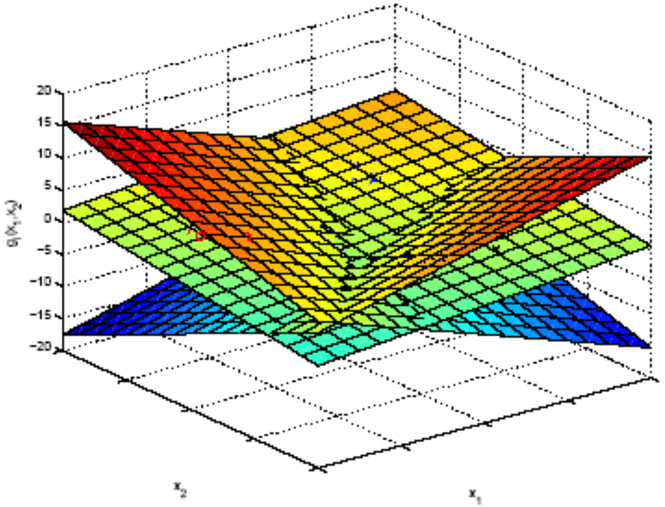
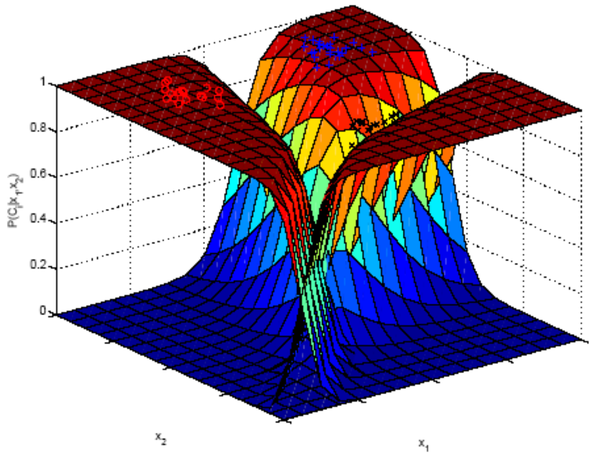


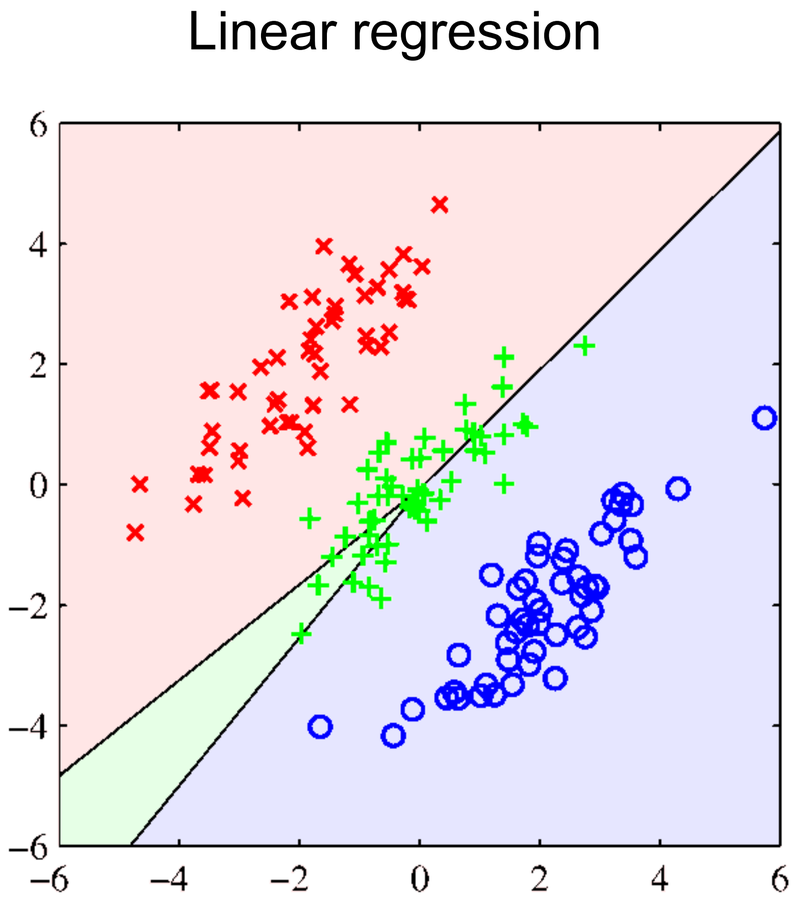
Discriminants (inputs)



Softmax (outputs)

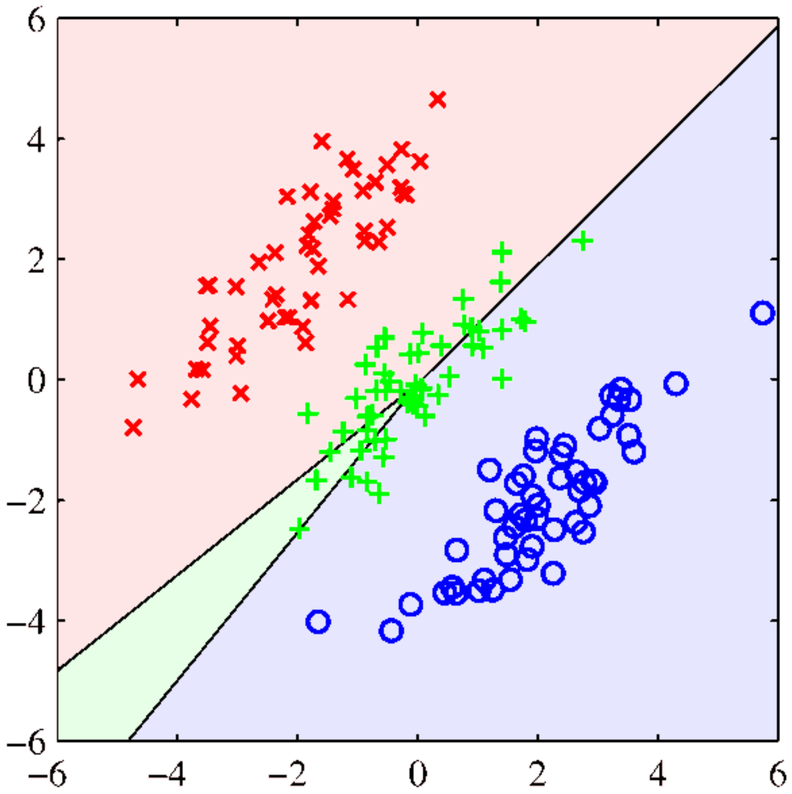
Logistic vs Linear Regression, n>2 classes

Logistic vs Linear Regression, n>2 classes

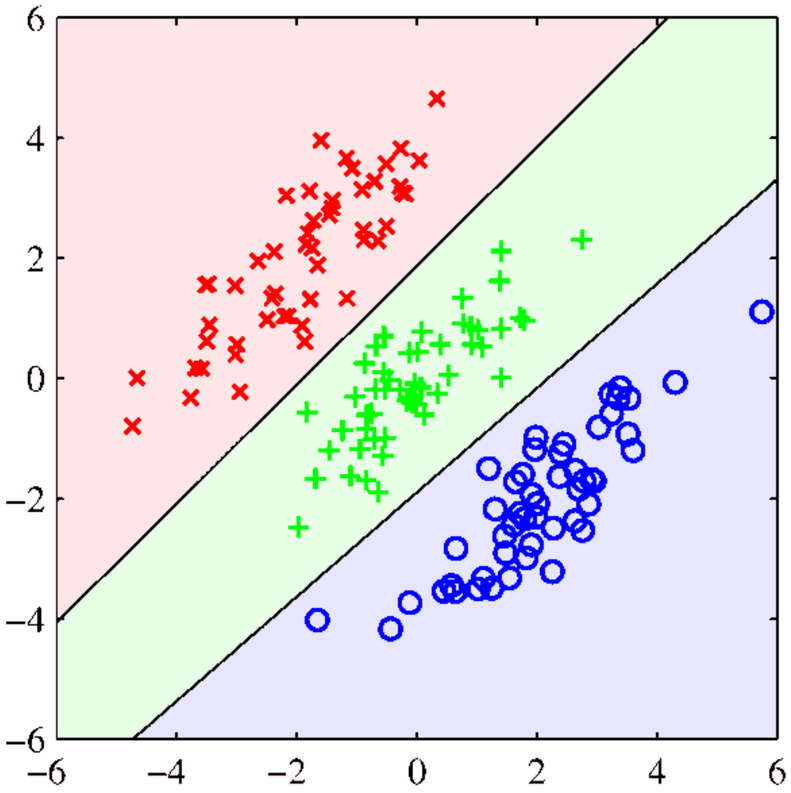


Logistic vs Linear Regression, n>2 classes

Linear regression

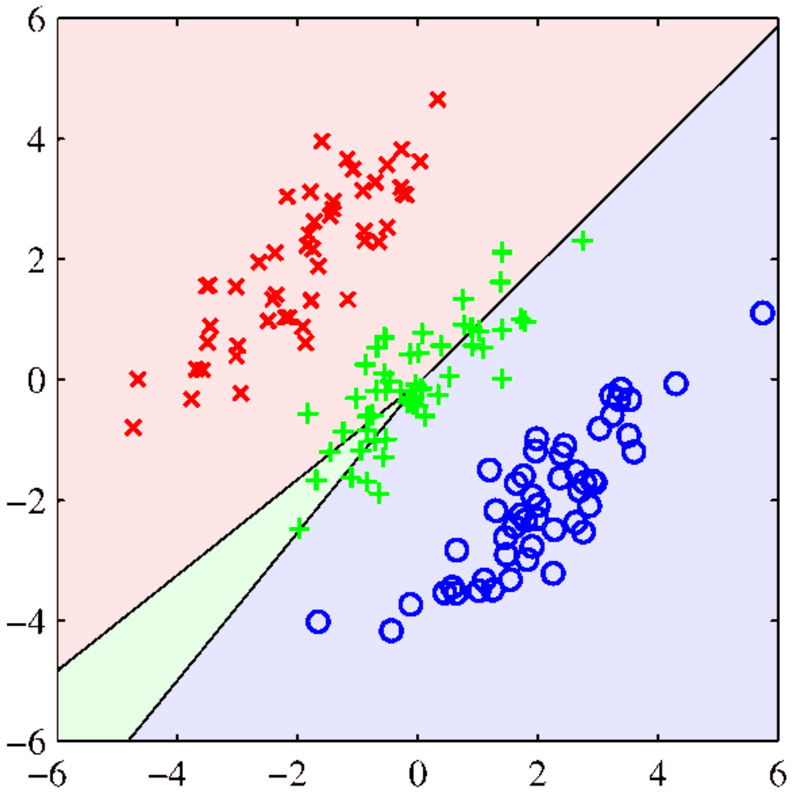


Logistic regression

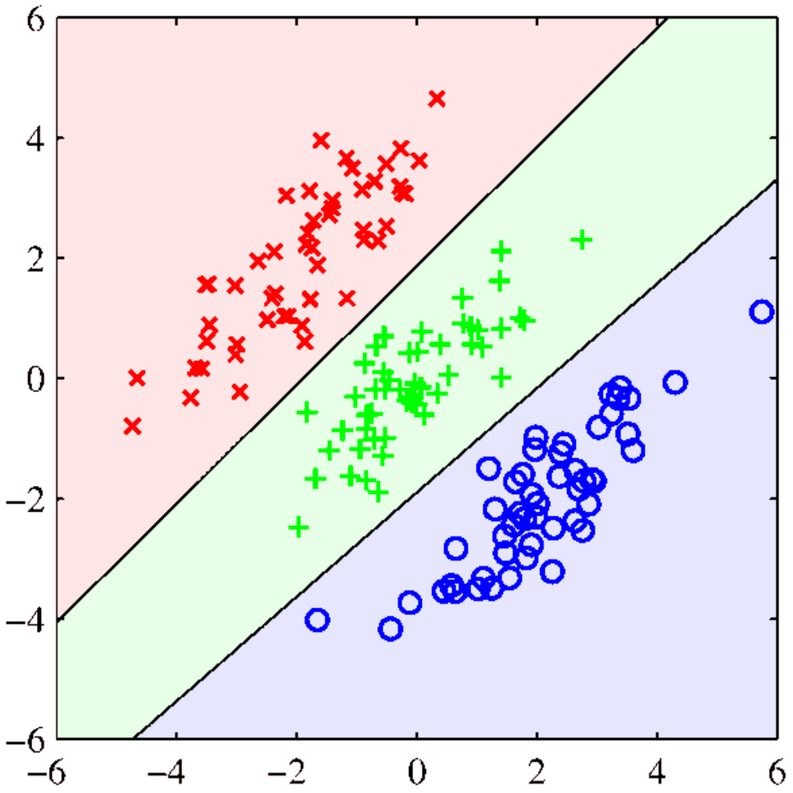


Logistic vs Linear Regression, n>2 classes

Linear regression



Logistic regression



Logistic regression does not exhibit the masking problem

LS Solution (in vector form)

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$$L(\mathbf{w}) = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon}$$

LS Solution (in vector form)

$$L(\mathbf{w}) = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon}$$
$$= (\mathbf{y} - \mathbf{X}\mathbf{w})^T (\mathbf{y} - \mathbf{X}\mathbf{w})$$

LS Solution (in vector form)

$$
\begin{aligned}
L(\mathbf{w}) &= \boldsymbol{\epsilon}^T \boldsymbol{\epsilon} \\
&= (\mathbf{y} - \mathbf{X}\mathbf{w})^T (\mathbf{y} - \mathbf{X}\mathbf{w}) \\
&= \mathbf{y}^T\mathbf{y} - 2\mathbf{y}^T\mathbf{X}\mathbf{w} + \mathbf{w}^T\mathbf{X}^T\mathbf{X}\mathbf{w}
\end{aligned}
$$

LS Solution (in vector form)

$$
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$$

Condition for minimum:

$$
\nabla L(\mathbf{w}^*) = \mathbf{0}
$$

LS Solution (in vector form)

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$$

Condition for minimum:

$$
\begin{aligned}
\nabla L(\mathbf{w}^*) &= \mathbf{0} \\
-2\mathbf{X}^T\mathbf{y} + 2\mathbf{X}^T\mathbf{X}\mathbf{w}^* &= \mathbf{0}
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$$

LS Solution (in vector form)

$$L(\mathbf{w}) = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon}$$
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Condition for minimum:

$$\nabla L(\mathbf{w}^*) = \mathbf{0}$$
$$-2\mathbf{X}^T\mathbf{y} + 2\mathbf{X}^T\mathbf{X}\mathbf{w}^* = \mathbf{0}$$
$$\mathbf{w}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

Gradient of Cross-entropy Loss

$$L(\mathbf{w}) = -\sum_{i=1}^{N} y^i \log g(\mathbf{w}^T \mathbf{x}^i) + (1 - y^i) \log(1 - g(\mathbf{w}^T \mathbf{x}^i))$$

Gradient of Cross-entropy Loss

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$$\frac{\partial L(\mathbf{w})}{\partial w_k} = -\sum_{i=1}^{N}\left[y^i \frac{1}{g(\mathbf{w}^T\mathbf{x}^i)}\frac{\partial g(\mathbf{w}^T\mathbf{x}^i)}{\partial w_k} + (1-y^i)\frac{1}{1-g(\mathbf{w}^T\mathbf{x}^i)}(-\frac{\partial g(\mathbf{w}^T\mathbf{x}^i)}{\partial w_k})\right]$$

using

$$g(x) = \frac{1}{1+\exp(-x)} \rightarrow \frac{dg}{dx} = g(x)(1-g(x))$$

$$= -\sum_{i=1}^{N}\left[y^i \frac{1}{g(\mathbf{w}^T\mathbf{x}^i)} - (1-y^i)\frac{1}{1-g(\mathbf{w}^T\mathbf{x}^i)}\right] g(\mathbf{w}^T\mathbf{x}^i)(1-g(\mathbf{w}^T\mathbf{x}^i))\frac{\partial \mathbf{w}^T\mathbf{x}^i}{\partial w_k}$$

$$= -\sum_{i=1}^{N}\left[y^i(1-g(\mathbf{w}^T\mathbf{x}^i)) - (1-y^i)g(\mathbf{w}^T\mathbf{x}^i)\right] x_k^i$$

$$= -\sum_{i=1}^{N}\left[y^i - g(\mathbf{w}^T\mathbf{x}^i)\right] \mathbf{x}_k^i$$

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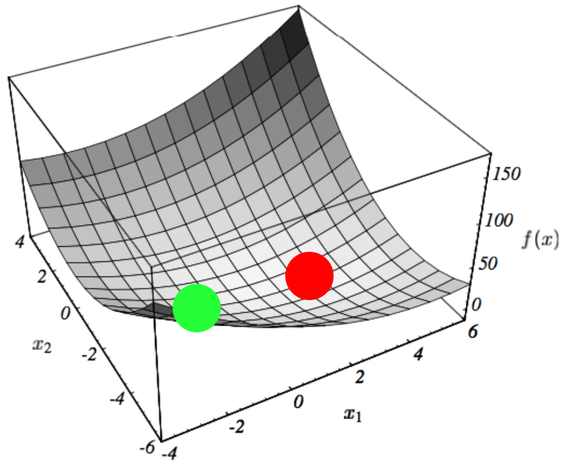
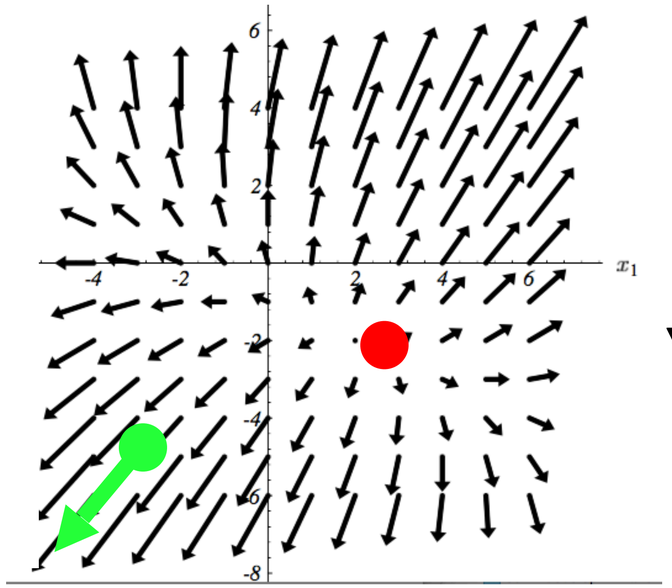
$$\nabla L(\mathbf{w}^*) = \mathbf{0}$$

nonlinear system of equations!!

Gradient Descent Minimization

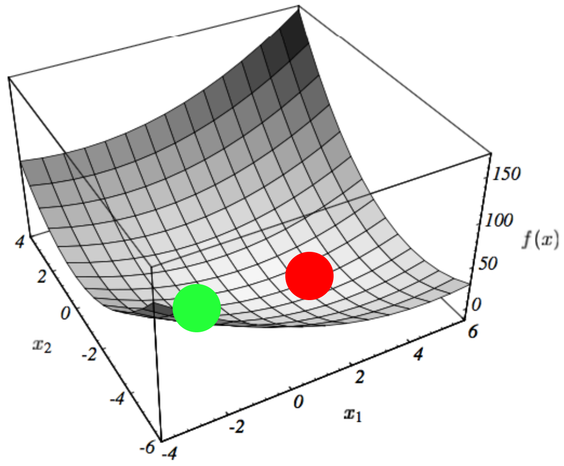


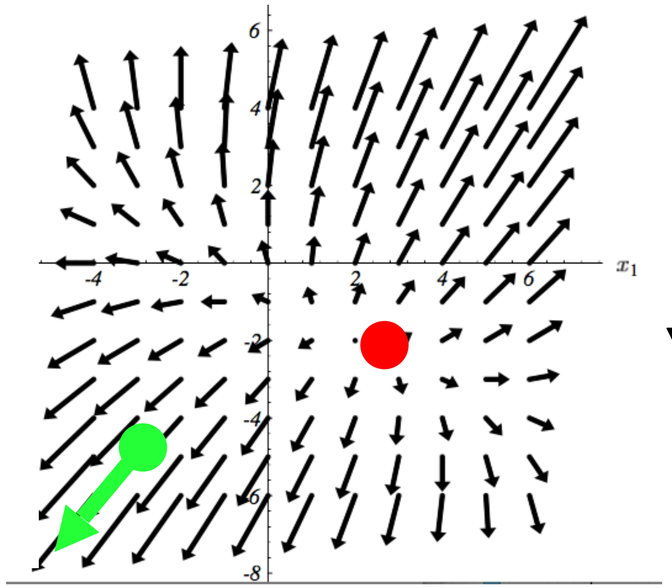


$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

Fact: gradient at any point gives direction of fastest increase

Gradient Descent Minimization





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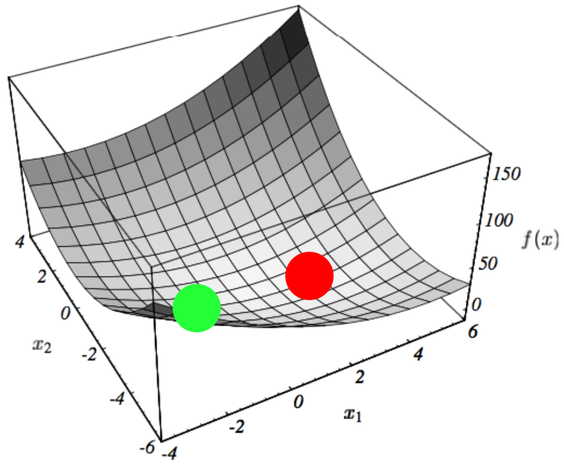
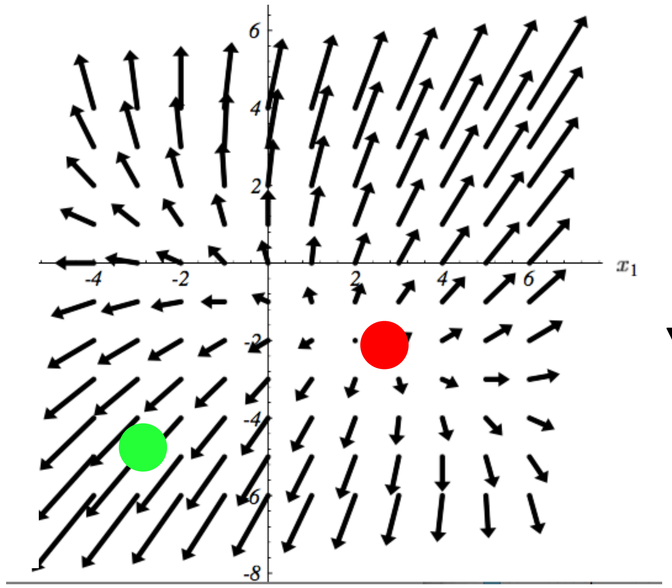
Idea: start at a point and move in the direction opposite to the gradient

Gradient Descent Minimization





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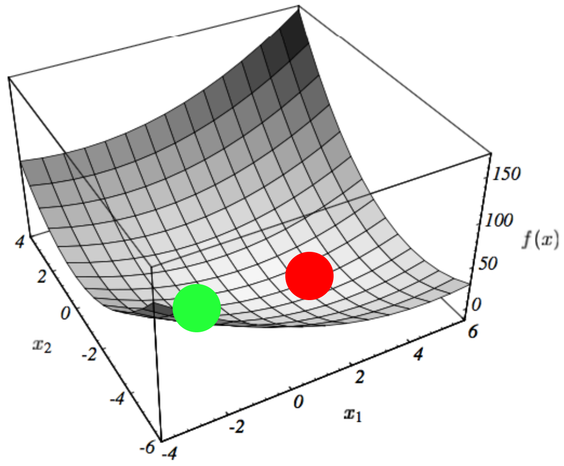

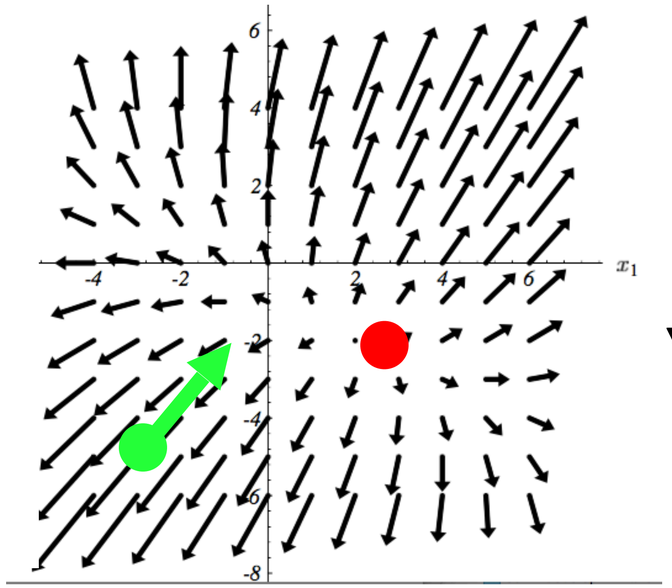

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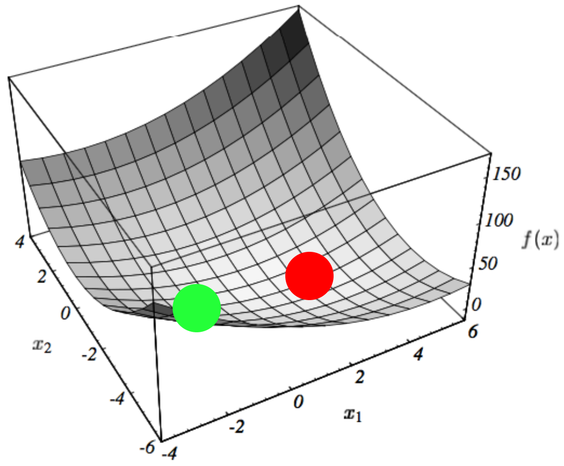

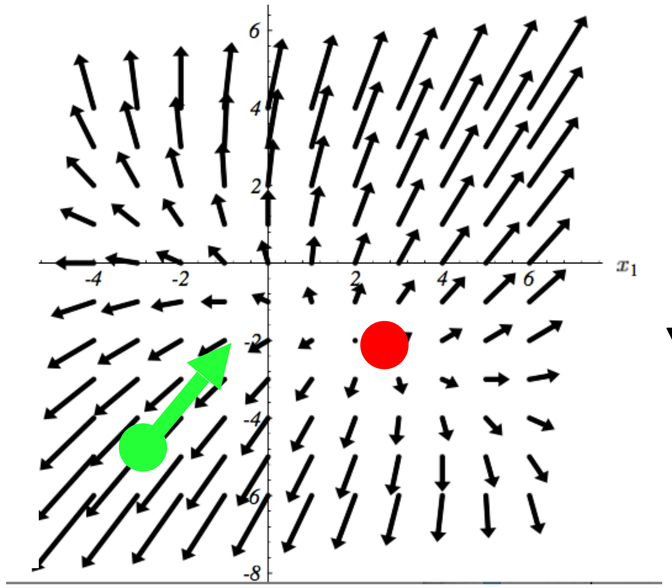

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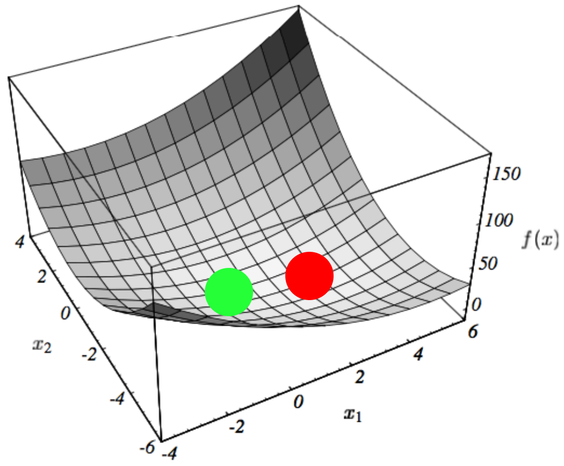
Fact: gradient at any point gives direction of fastest increase

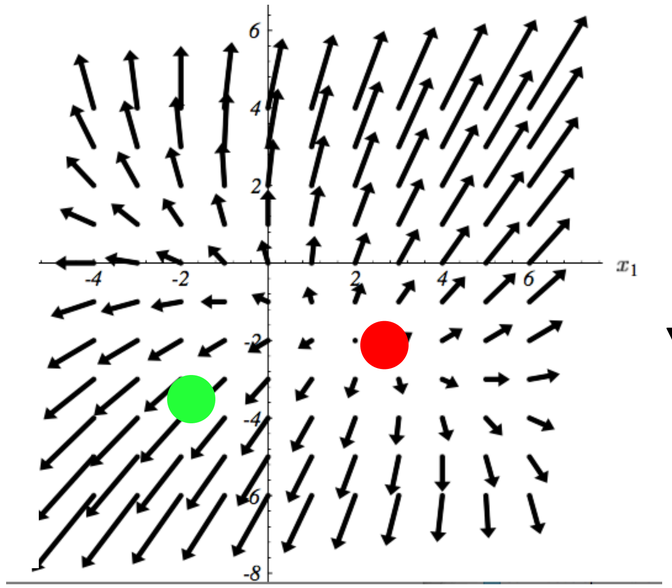
Idea: start at a point and move in the direction opposite to the gradient

Initialize: $\mathbf{x}_0$

Update: $\mathbf{x}_{i+1} = \mathbf{x}_i - \alpha \nabla f(\mathbf{x}_i)$ **i=0**

Gradient Descent Minimization





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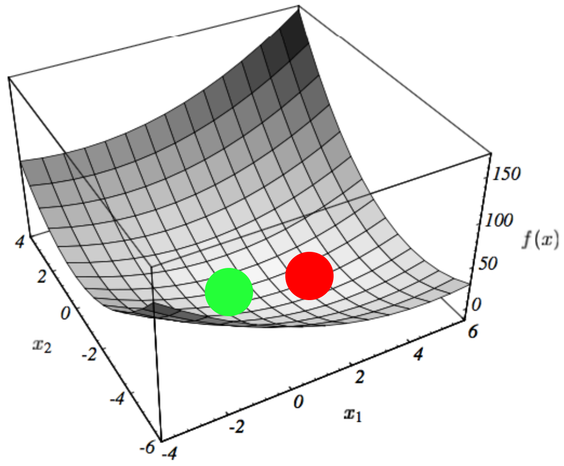
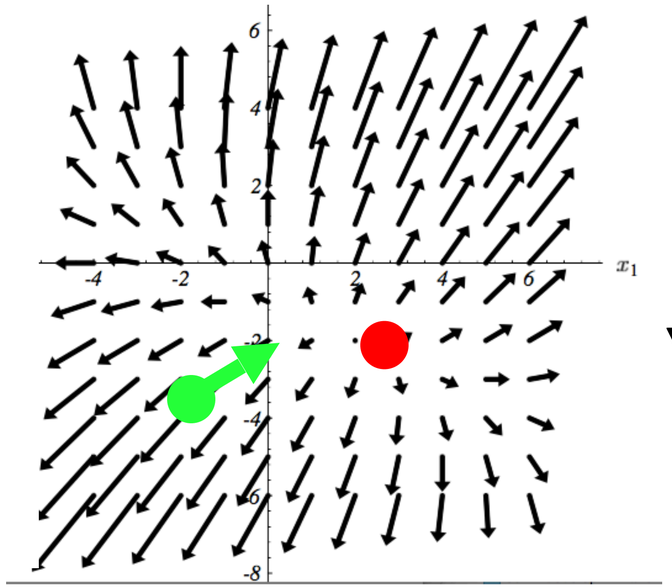
Gradient Descent Minimization





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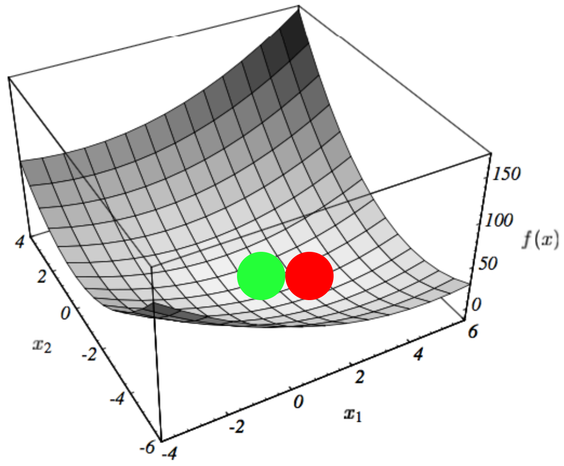
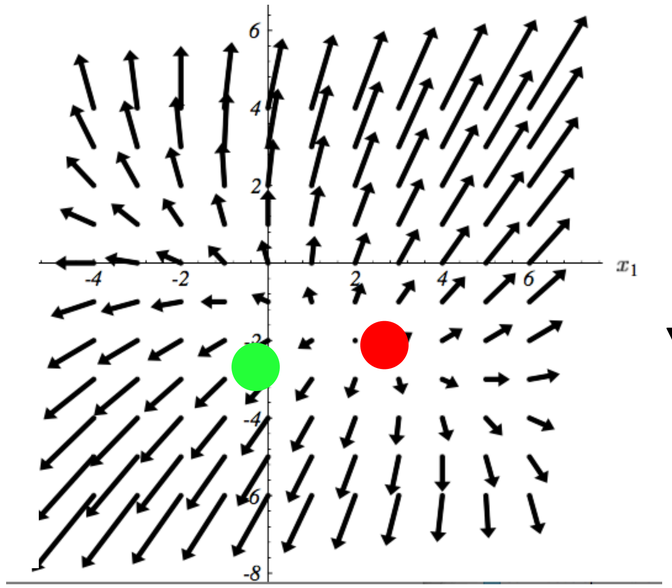
Gradient Descent Minimization





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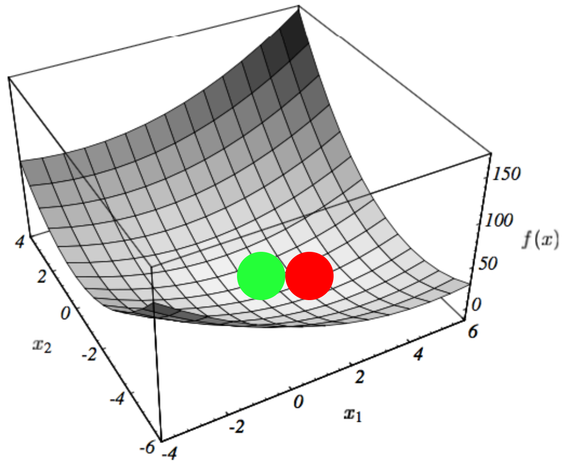
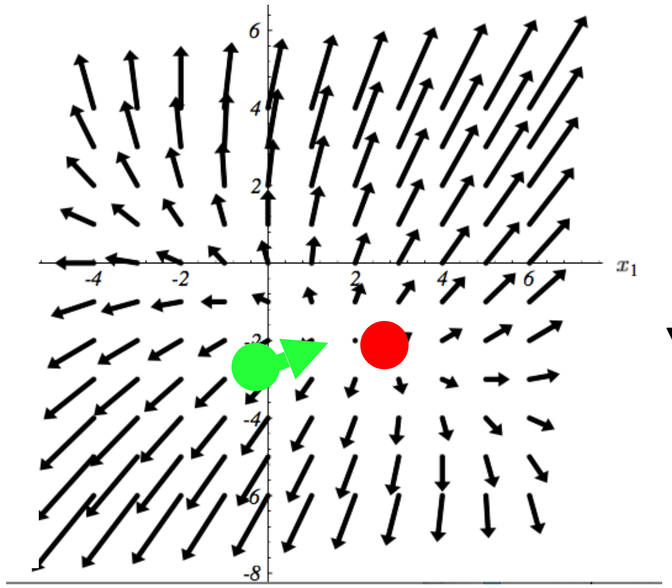
Update: $$\mathbf{x}_{i+1} = \mathbf{x}_i - \alpha \nabla f(\mathbf{x}_i)$$ **i=2**

Gradient Descent Minimization





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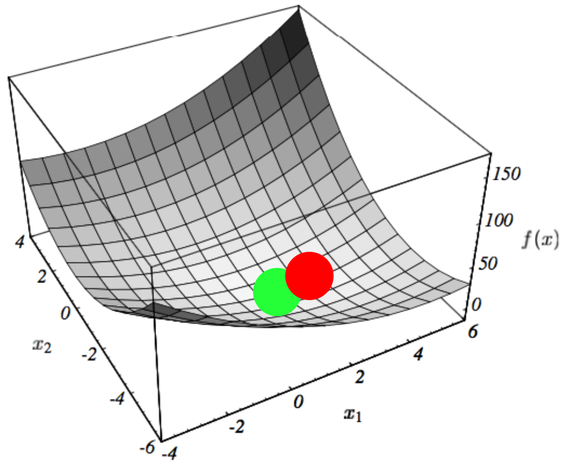
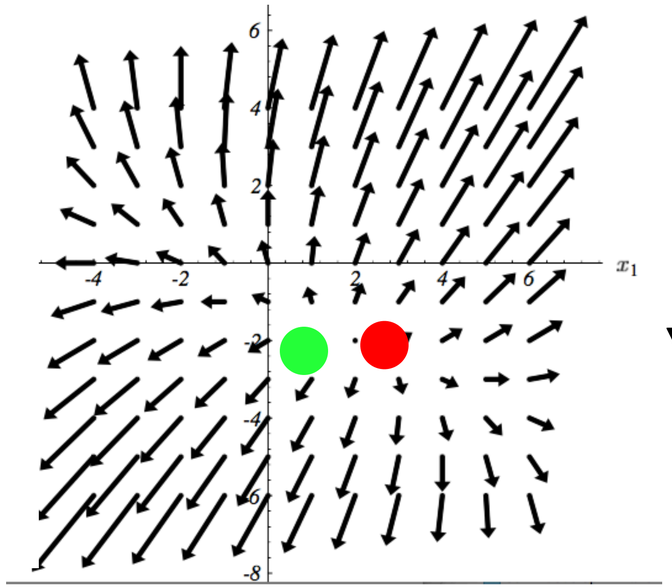
Gradient Descent Minimization





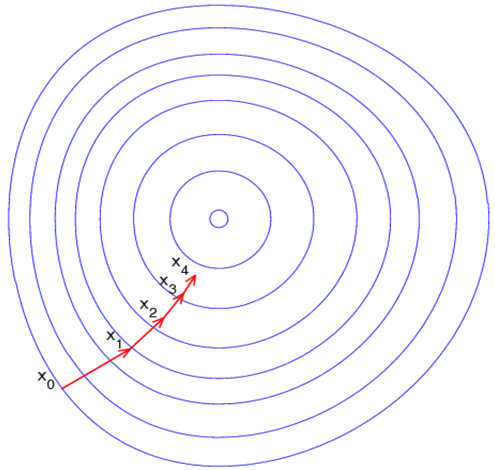
$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

Update: $$\mathbf{x}_{i+1} = \mathbf{x}_i - \alpha \nabla f(\mathbf{x}_i)$$ **i=3**

Gradient Descent Minimization

Initialize: $\mathbf{x}_0$

Update: $\mathbf{x}_{i+1} = \mathbf{x}_i - \alpha \nabla f(\mathbf{x}_i)$

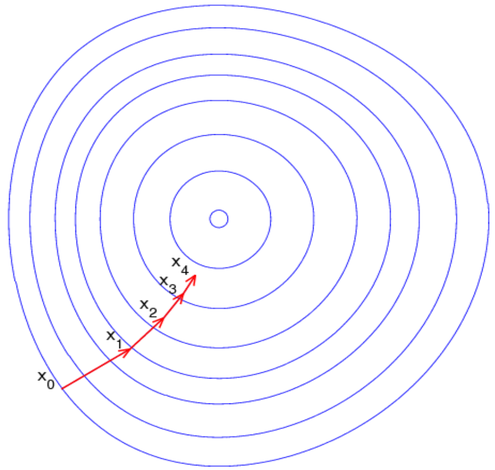


Gradient Descent Minimization

Initialize: $\mathbf{x}_0$

Update: $\mathbf{x}_{i+1} = \mathbf{x}_i - \alpha \nabla f(\mathbf{x}_i)$

We can always make it converge for a convex function.



Gradient Descent Minimization

Initialize: $\mathbf{x}_0$

Update: $\mathbf{x}_{i+1} = \mathbf{x}_i - \alpha \nabla f(\mathbf{x}_i)$

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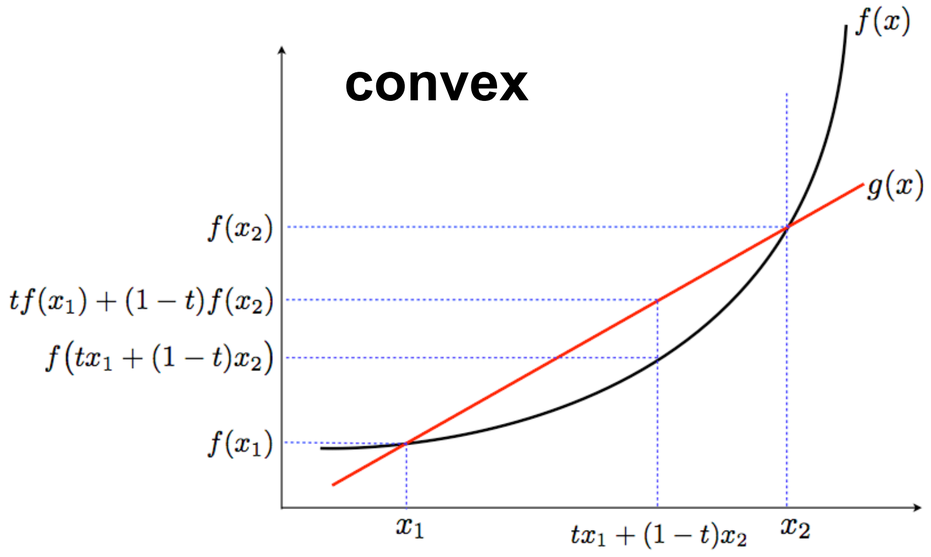

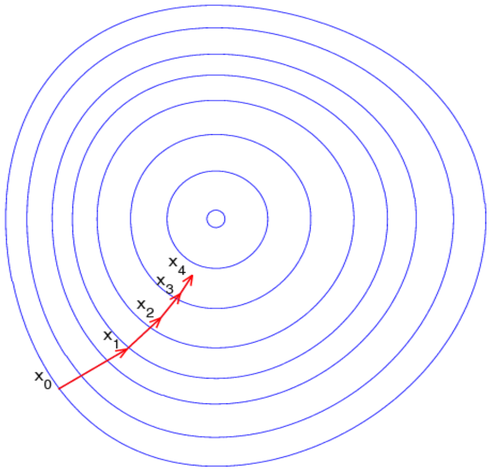

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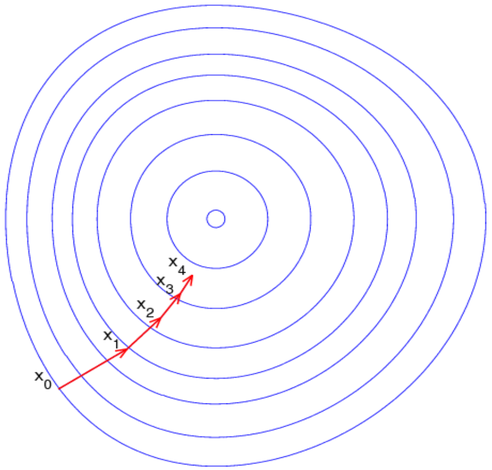

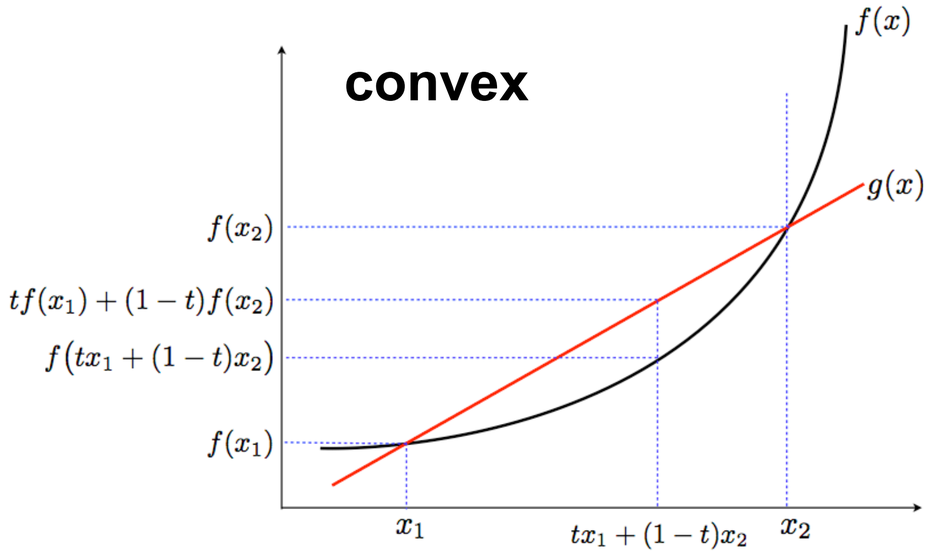

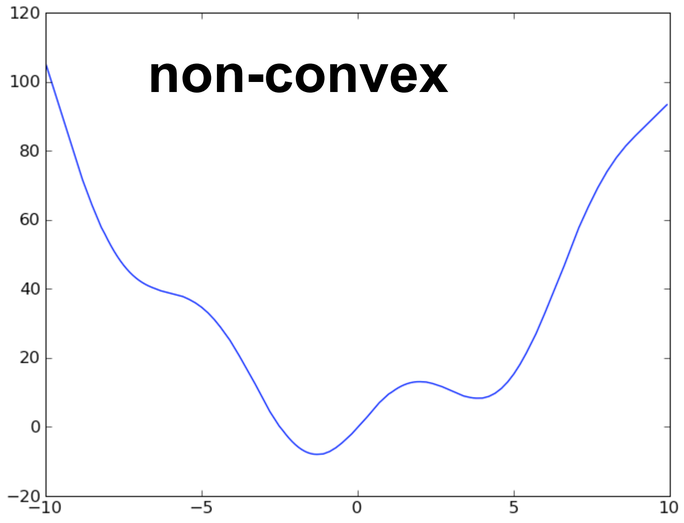

XOR Problem

XOR Problem

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

XOR Problem

$$y = f(x_1, x_2)$$

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

XOR Problem

$$y = f(x_1, x_2)$$

x_1	x_2	y
0	0	0
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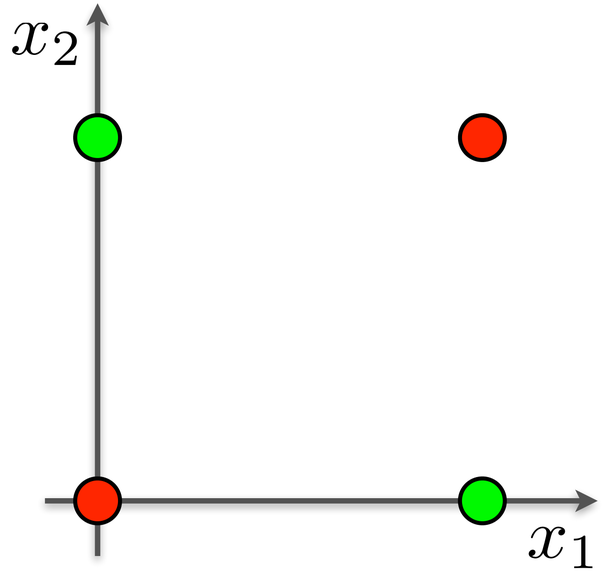

XOR Problem

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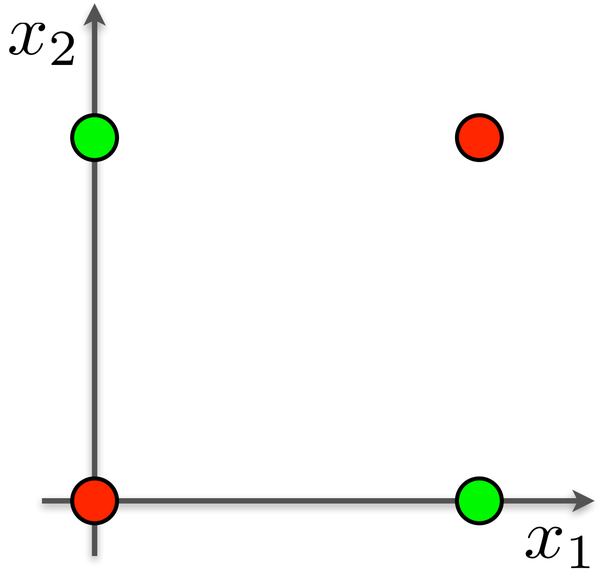
x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

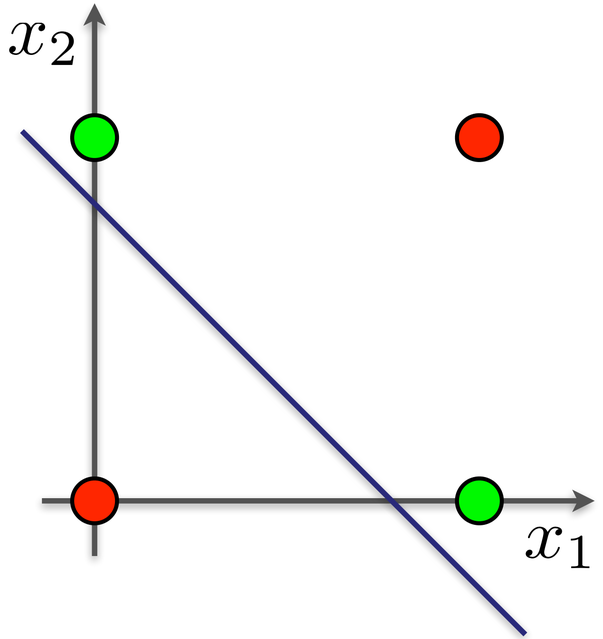


$$y = f(w_0, w_1, w_2) = \mathcal{H}(w_0 + w_1 x_1 + w_2 x_2)$$

XOR Problem

$$y = f(x_1, x_2)$$

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

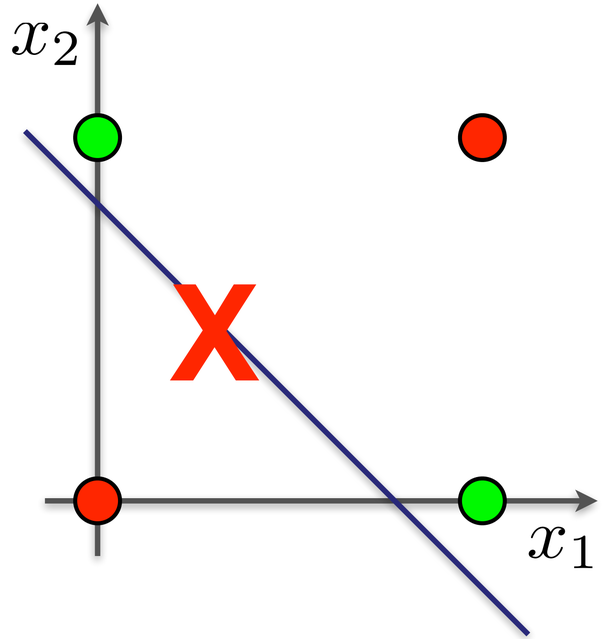


$$y = f(w_0, w_1, w_2) = \mathcal{H}(w_0 + w_1 x_1 + w_2 x_2)$$

XOR Problem

$$y = f(x_1, x_2)$$

x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0

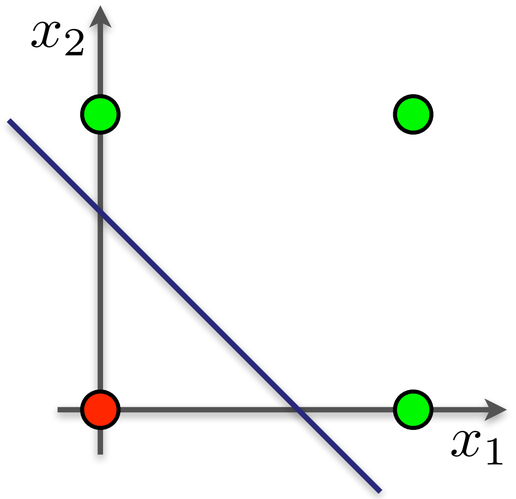


$$y = f(w_0, w_1, w_2) = \mathcal{H}(w_0 + w_1 x_1 + w_2 x_2)$$

XOR Problem

XOR Problem

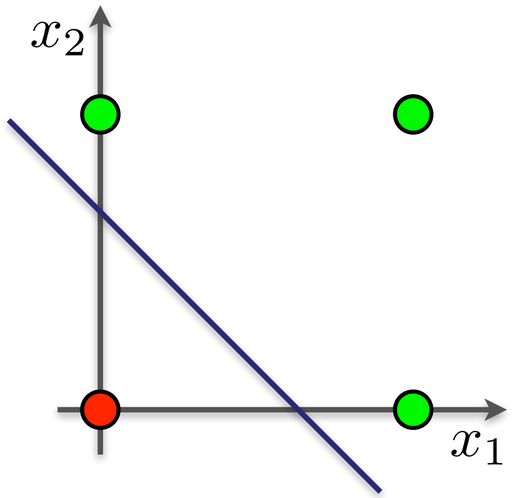
x_1	x_2	z_1
0	0	0
0	1	1
1	0	1
1	1	1

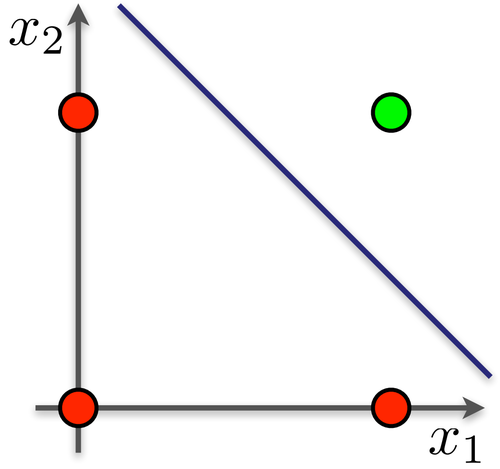


XOR Problem

x_1	x_2	z_1
0	0	0
0	1	1
1	0	1
1	1	1

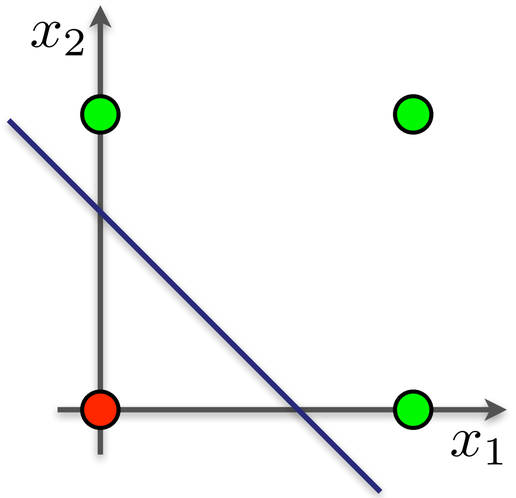
x_1	x_2	z_2
0	0	0
0	1	0
1	0	0
1	1	1



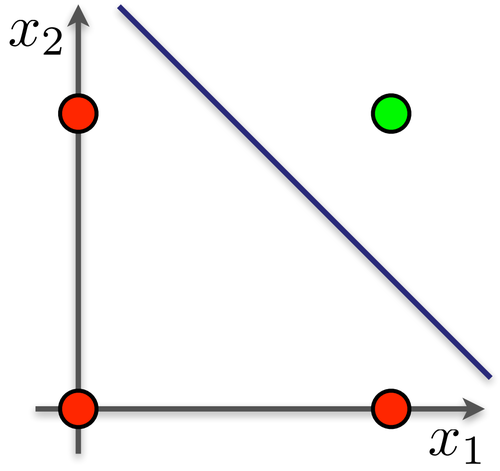


XOR Problem

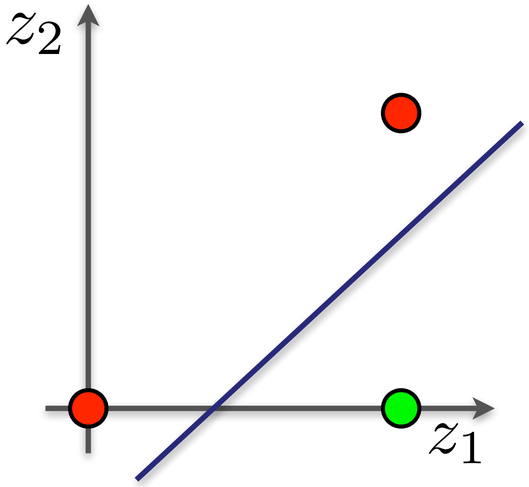
x_1	x_2	z_1
0	0	0
0	1	1
1	0	1
1	1	1



x_1	x_2	z_2
0	0	0
0	1	0
1	0	0
1	1	1



z_1	z_2	y
0	0	0
1	0	1
1	0	1
1	1	0



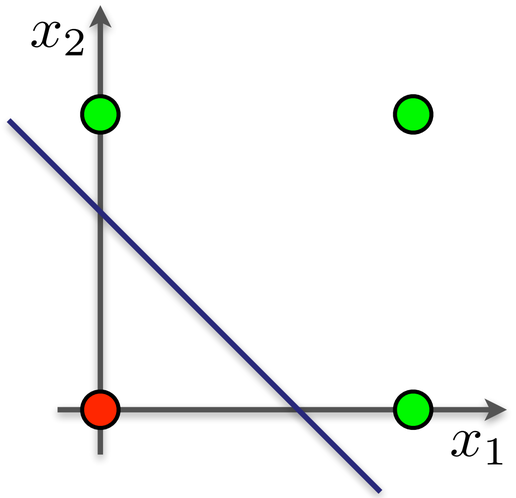
XOR Problem

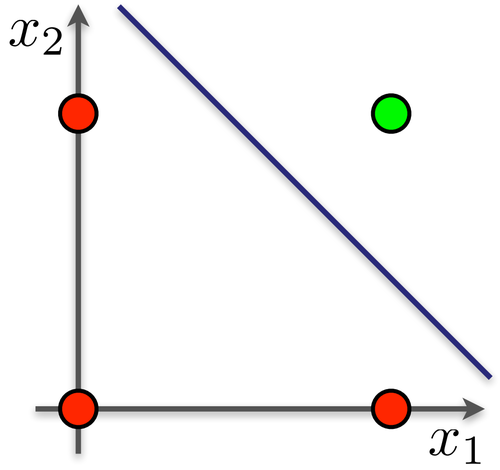
$$y = f(z_1, z_2) = f(g_1(x_1, x_2), g_2(x_1, x_2))$$

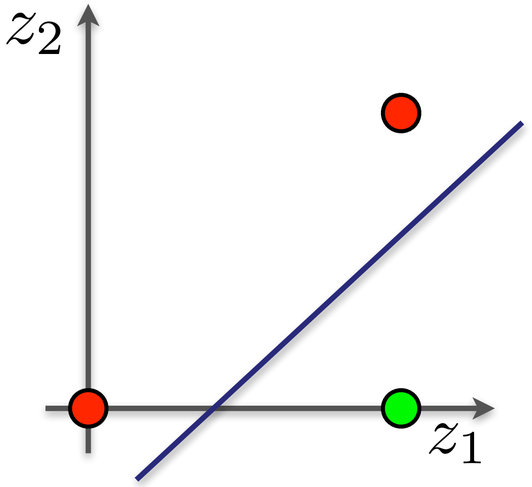
x_1	x_2	z_1
0	0	0
0	1	1
1	0	1
1	1	1

x_1	x_2	z_2
0	0	0
0	1	0
1	0	0
1	1	1

z_1	z_2	y
0	0	0
1	0	1
1	0	1
1	1	0

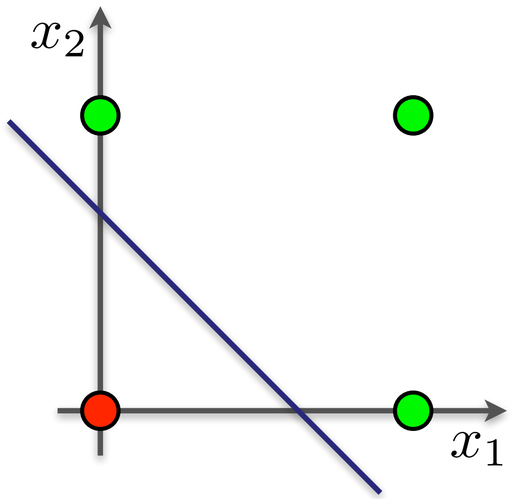


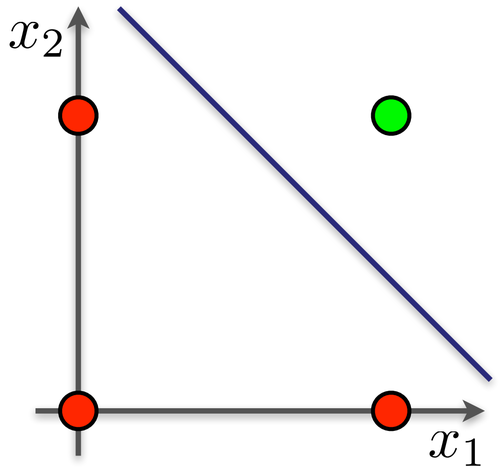


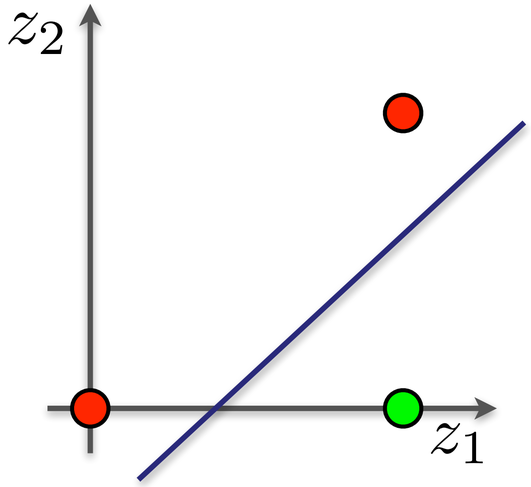


XOR Problem

$$y = f(z_1, z_2) = f(g_1(x_1, x_2), g_2(x_1, x_2))$$

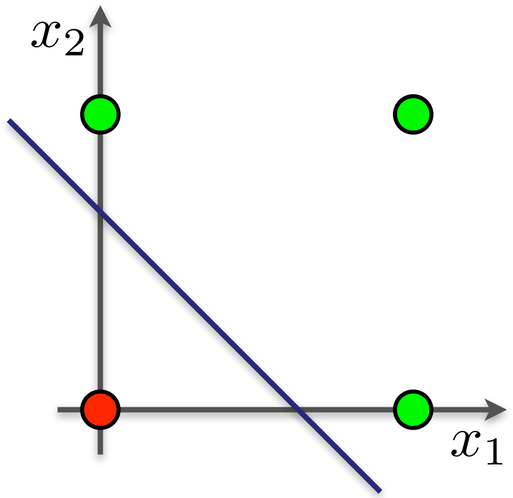


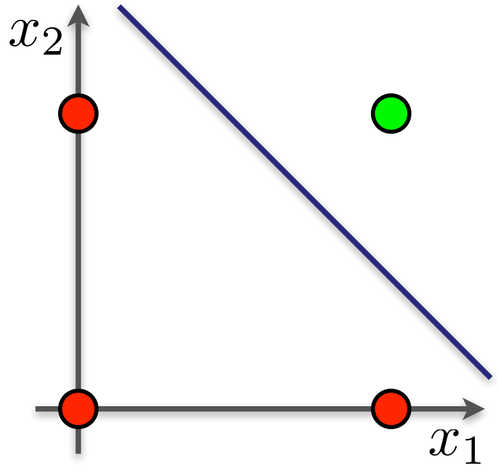


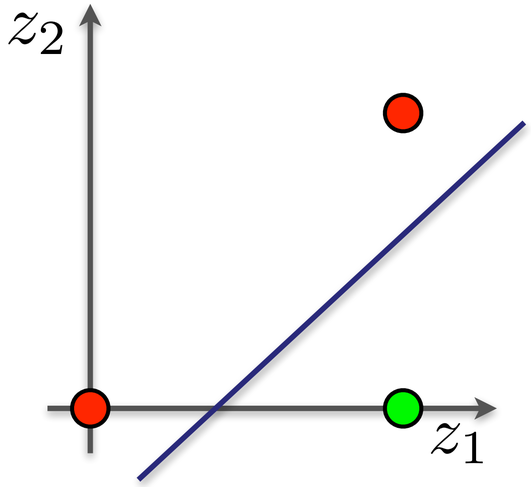


XOR Problem

$$y = f(z_1, z_2) = f(g_1(x_1, x_2), g_2(x_1, x_2))$$
$$= f\left(\mathcal{H}(\mathbf{w}^1, x_1, x_2), \mathcal{H}(\mathbf{w}^2, x_1, x_2)\right)$$

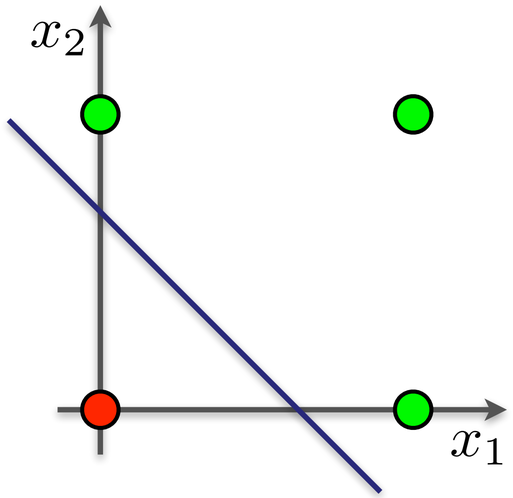


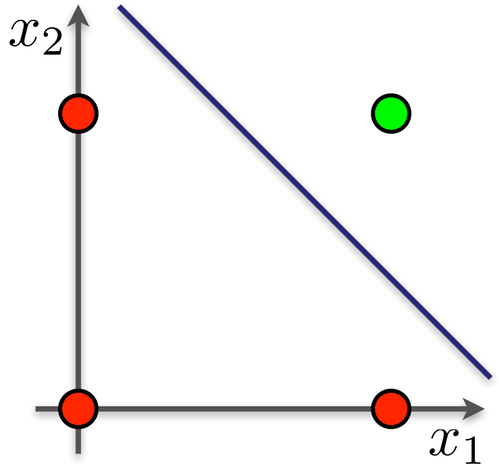


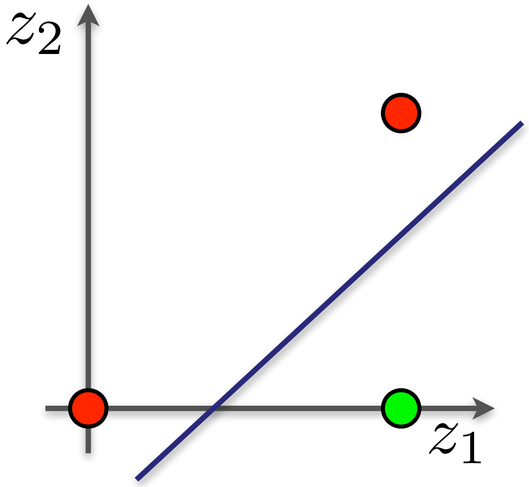


XOR Problem

$$
\begin{aligned}
y = f(z_1, z_2) &= f(g_1(x_1, x_2), g_2(x_1, x_2)) \\
&= f\left(\mathcal{H}(\mathbf{w}^1, x_1, x_2), \mathcal{H}(\mathbf{w}^2, x_1, x_2)\right) \\
&= \mathcal{H}(\mathbf{w}^3, \mathcal{H}(g_1(\mathbf{w}^1, x_1, x_2)), \mathcal{H}(g_2(\mathbf{w}^2, x_1, x_2))
\end{aligned}
$$

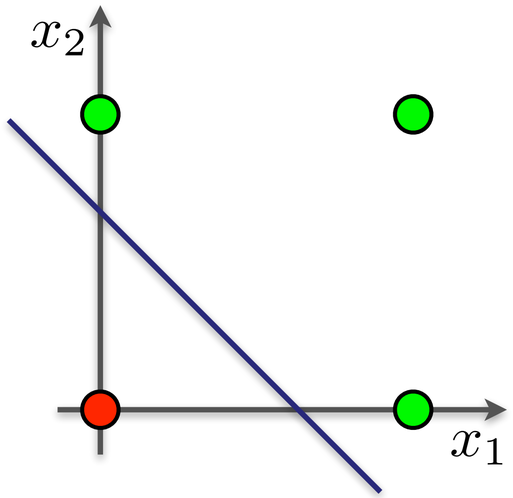


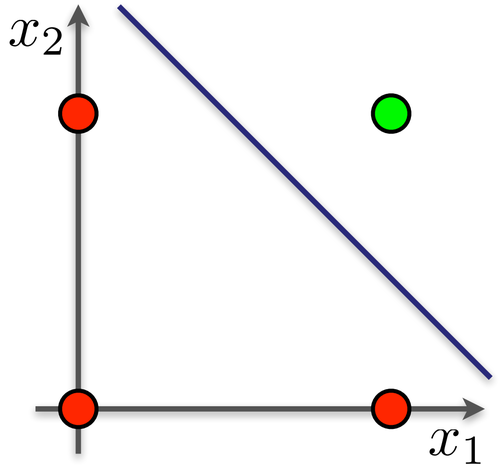


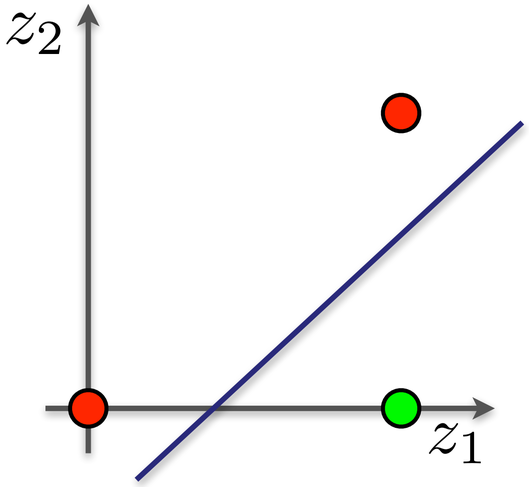


XOR Problem

$$
\begin{aligned}
y = f(z_1, z_2) &= f(g_1(x_1, x_2), g_2(x_1, x_2)) \\
&= f\left(\mathcal{H}(\mathbf{w}^1, x_1, x_2), \mathcal{H}(\mathbf{w}^2, x_1, x_2)\right) \\
&= \mathcal{H}(\mathbf{w}^3, \mathcal{H}(g_1(\mathbf{w}^1, x_1, x_2)), \mathcal{H}(g_2(\mathbf{w}^2, x_1, x_2)))
\end{aligned}
$$







XOR Problem

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\begin{aligned}
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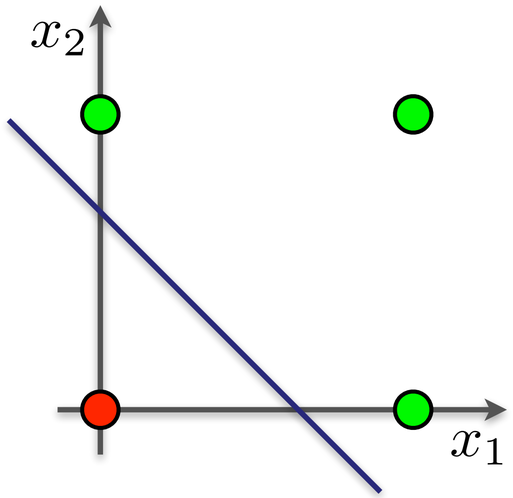
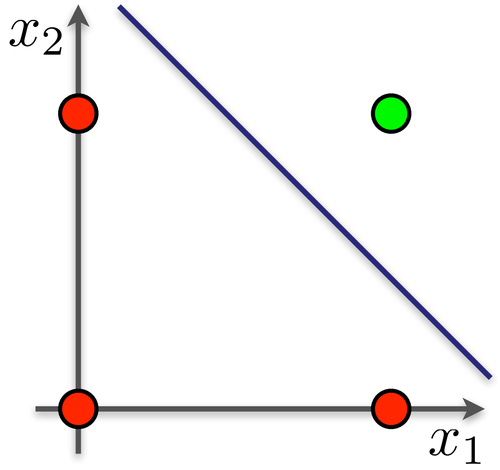
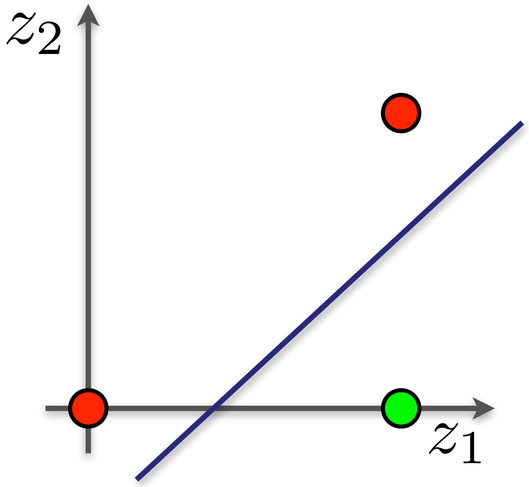
&= \mathcal{H}(\mathbf{w}^3, \mathcal{H}(g_1(\mathbf{w}^1, x_1, x_2)), \mathcal{H}(g_2(\mathbf{w}^2, x_1, x_2)))
\end{aligned}
$$







XOR Problem

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\begin{aligned}
y = f(z_1, z_2) &= f(g_1(x_1, x_2), g_2(x_1, x_2)) \\
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XOR Problem

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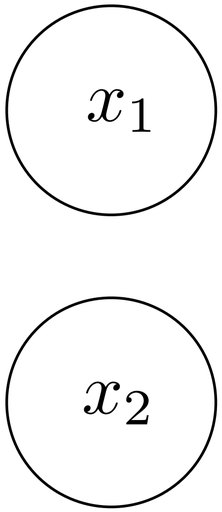
\end{aligned}
$$



XOR Problem

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\begin{aligned}
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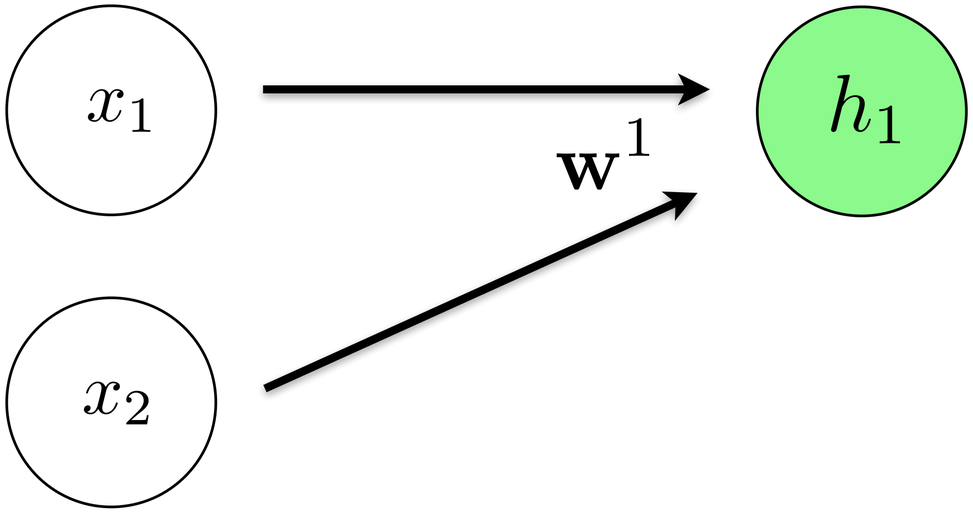
&= \mathcal{H}(\mathbf{w}^3, \mathcal{H}(g_1(\mathbf{w}^1, x_1, x_2)), \mathcal{H}(g_2(\mathbf{w}^2, x_1, x_2)))
\end{aligned}
$$



XOR Problem

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\begin{aligned}
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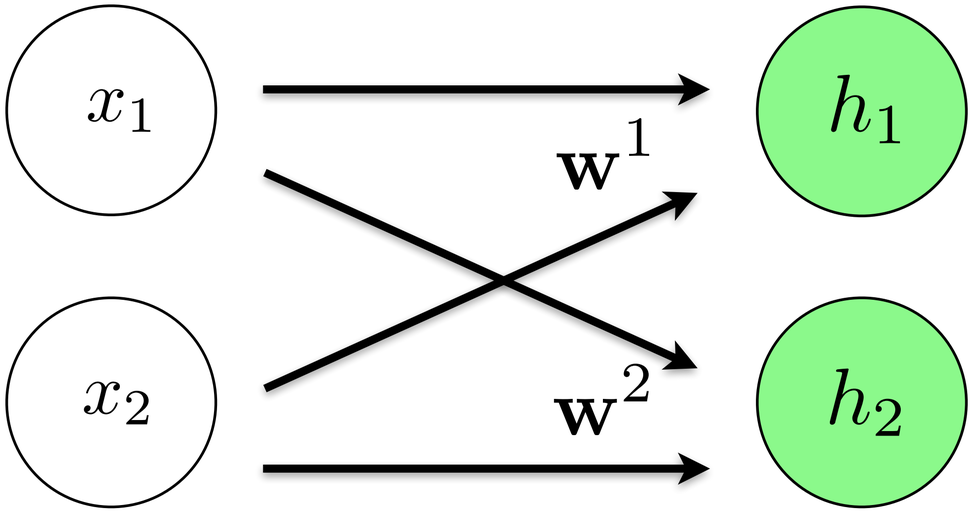
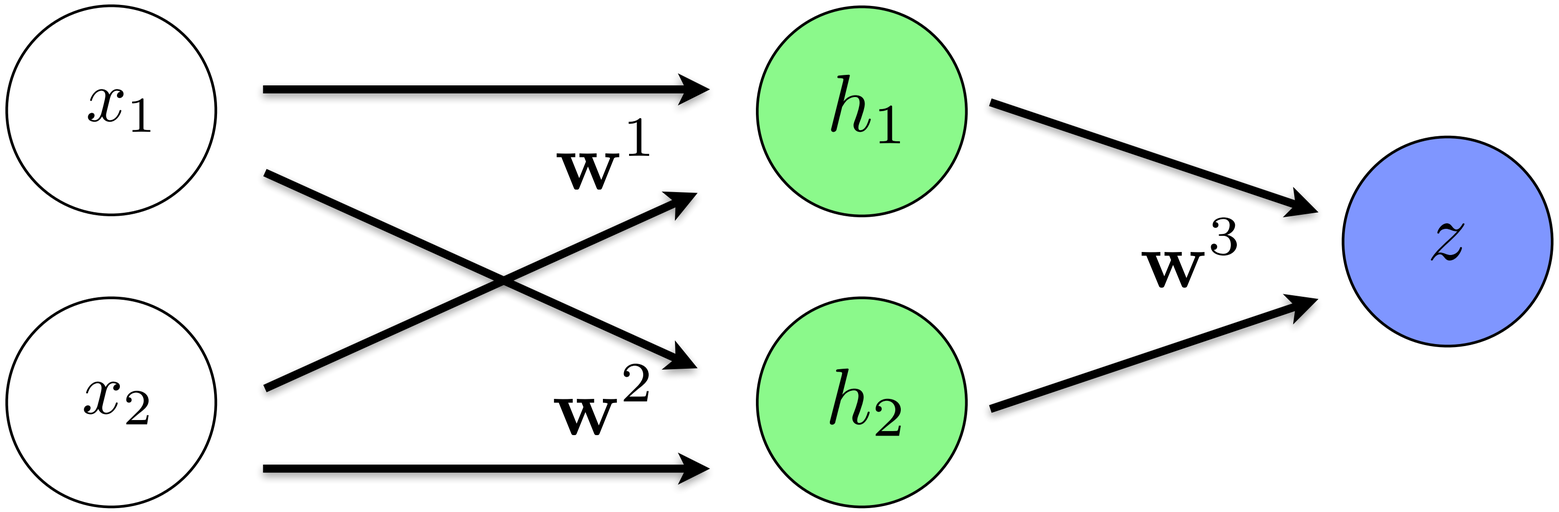
&= \mathcal{H}(\mathbf{w}^3, \mathcal{H}(g_1(\mathbf{w}^1, x_1, x_2)), \mathcal{H}(g_2(\mathbf{w}^2, x_1, x_2)))
\end{aligned}
$$

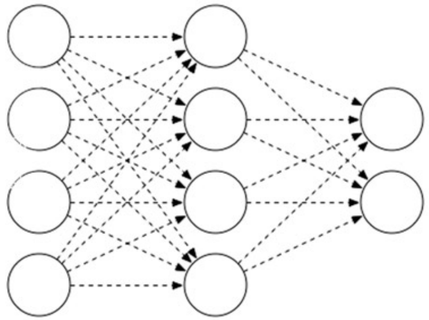


XOR Problem

$$
\begin{aligned}
y = f(z_1, z_2) &= f(g_1(x_1, x_2), g_2(x_1, x_2)) \\
&= f\left(\mathcal{H}(\mathbf{w}^1, x_1, x_2), \mathcal{H}(\mathbf{w}^2, x_1, x_2)\right) \\
&= \mathcal{H}(\mathbf{w}^3, \mathcal{H}(g_1(\mathbf{w}^1, x_1, x_2)), \mathcal{H}(g_2(\mathbf{w}^2, x_1, x_2)))
\end{aligned}
$$



Course Information (slides/code/comments)



http://geometry.cs.ucl.ac.uk/dl4g/